SLICE FILTRATION AND TORSION THEORY IN MOTIVIC COHOMOLOGY

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ABSTRACT OF THE DISSERTATION

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We show that the category **HI** of homotopy invariant Nisnevich sheaves with transfers and the category **CycMod** are each equipped with a strong filtrations and a strong cofiltration. To do so, we first define pre-coradicals and coradicals on well-powered abelian categories, and show that every isomorphism class of coradical is associated to a canonical torsion theory. We then summarize the theory of motivic cohomology needed to define **HI**, its symmetric monoidal structure $\otimes^H$ and its partial internal hom $\underline{\mathrm{Hom}}_{\mathbf{HI}}$. Along the way, we recall the construction of the slice filtration on $\mathbf{DM}^{\mathrm{eff},-}$, and extend the filtration structure on $\mathbf{DM}^{\mathrm{eff},-}$ to **DM**.

We then define and construct the torsion filtration on **HI** by constructing a sequence of coradicals. We explain how the torsion filtration is related to the slice filtration on $\mathbf{DM}^{\mathrm{eff},-}$. We extend the torsion filtration to the category $\mathbf{HI}_*$ of homotopic modules. Appealing to the categorical equivalence between $\mathbf{HI}_*$ and **CycMod**, we obtain the torsion filtration on **CycMod**. Finally, we generalize the conditions under which torsion filtrations exist for the heart of a tensor triangulated category.

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Dedication

To my high school math teacher, John Reutershan.

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Chapter 1

Introduction

The goal of this thesis is to show that the abelian categories **HI** of homotopy invariant Nisnevich sheaves with transfers and **CycMod** of Rost's cycle modules admit two filtrations. Here, a (weak) filtration of a category roughly means a nested sequence of subcategories together with reflection or coreflection functors from the category to each of its subcategories. The filtrations are induced by the slice filtration on the tensor triangulated category $\mathbf{DM}^{\text{eff},-}$ of Voevodsky's derived category of motive. One of the key ingredients in constructing the three filtrations is a sequence of adjoint functors from **HI** to itself, coming from the triangulated tensor structure of $\mathbf{DM}^{\text{eff},-}$. The other key ingredient is torsion theory.

We first revisit the basic definition and results of classical torsion theory for well-powered abelian categories, as developed in [BJV] or [Dic66] (Chapter 2). However, instead of focusing on the relationship between torsion theories and radicals, we introduce the theory from the perspective of coradicals, which are radicals in the opposite category.

We then summarize the theory in motivic cohomology needed to understand the tensor triangulated structure on $\mathbf{DM}^{\text{eff},-}$ (Chapters 3 and 4). These are taken from early lectures in [MVW]. The main results that we highlight in these two chapters are the Cancellation Theorem of Voevodsky and the existence of an object $\mathbb{Z}(1)$ of $\mathbf{DM}^{\text{eff},-}$ which gives rise to a pair of adjoint endofunctors on $\mathbf{DM}^{\text{eff},-}$.

These results provide the necessary scaffold to introduce the slice filtration on $\mathbf{DM}^{\text{eff},-}$. The term "slice filtration" is the name of a filtration structure on the stable homotopy category of motives that Voevodsky defined in [Voe02b]. The analogous structure for $\mathbf{DM}^{\text{eff},-}$ is constructed by Huber and Kahn in [HK06]. We summarize the main properties of the slice filtration on $\mathbf{DM}^{\mathrm{eff},-}$ in Chapter 5, and develop an extension of the slice filtration to the category **DM**, which is the triangulated category obtained from $\mathbf{DM}^{\mathrm{eff},-}$ by inverting the Tate motive (see Section 5.3).

In Chapter 6, we develop several filtrations on **HI**. We first note that $\mathbf{DM}^{\mathrm{eff},-}$ is equipped with a t-structure in the sense of [BBD], and that **HI** is categorically equivalent to the heart. An obvious question to ask is whether the filtration structure on $\mathbf{DM}^{\mathrm{eff},-}$ induces a similar structure on **HI**. In fact, the slice filtration on $\mathbf{DM}^{\mathrm{eff},-}$ does induce two filtrations on **HI**. In addition, the reflection functors from one of the filtrations define a sequence of coradicals. Applying the results of Chapter 2, we obtain a third filtration, which has the additional property that the filtration on **HI** induces a functorial filtration of each object of **HI**. We coin the term "torsion filtration" to describe the filtrations that come from a sequence of coradicals.

We then extend the torsion filtrations on **HI** to the abelian category $\mathbf{HI}_*$ of homotopy modules (Chapter 7). The key is to construct a $\mathbb{Z}$-indexed sequence of coradicals on $\mathbf{HI}_*$. Once we have accomplished this, applying the results of Chapter 2, we obtain filtrations of $\mathbf{HI}_*$. Using the fact that $\mathbf{HI}_*$ is categorically equivalent to **CycMod**, we conclude that these filtrations exist on **CycMod**.

In the last chapter, we summarize the results of the previous chapters by axiomatizing the conditions on a triangulated category with a t-structure such that the heart is equipped with a sequence of coradicals whose associated torsion theories form two filtrations on the category. The essential ingredient is for a tensor triangulated category with a t-structure to be equipped with a Tate object — an object S in the heart such that the functor given by tensoring with S admits a right adjoint — such that the Cancellation Theorem holds for S. We call such a triangulated category *torsion monoidal*, and we show that the heart of any torsion monoidal category is equipped with three filtrations, two of which are induced by a sequence of coradicals.

For the remainder of the thesis, we assume that k is a perfect field.

Chapter 2

Coradicals and Torsion Theory

In this chapter, we develop the basics of torsion theory in a categorical setting. The concepts and results here closely follow those of [BJV] and [Dic66], except we develop the theory from the dual perspective of coradicals. The ideas are not new; neither is the methodology. We have included proofs of all results in this chapter for the convenience of the reader.

2.1 Coradicals

For the remainder of the chapter, let $\mathscr{A}$ be a cocomplete well-powered abelian category. That is, $\mathscr{A}$ is closed under small direct sums, and for every object A in $\mathscr{A}$, the collection of subobjects of A forms a set.

Definition 2.1.1. For a given subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$, and an object A in $\mathscr{A}$, we say $A_{\mathscr{C}}$ is a *largest* $\mathscr{C}$*-subobject of* A if $A_{\mathscr{C}}$ is a subobject of A belonging to $\mathscr{C}$ such that for all subobjects B in $\mathscr{C}$, the monomorphism $B \hookrightarrow A$ factors through $A_{\mathscr{C}}$. That is, for every diagram

$$\begin{array}{ccc} & & B \\ & \swarrow^{f} & \big\downarrow{\scriptstyle i} \\ A_{\mathscr{C}} & \overset{j}{\hookrightarrow} & A \end{array}$$

where B is in $\mathscr{C}$, there exists a map $B \xrightarrow{f} A_{\mathscr{C}}$ such that $jf = i$.

We say a subcategory $\mathscr{C}$ of $\mathscr{A}$ is *reflective* (resp., *coreflective*) if the inclusion of $\mathscr{C}$ into $\mathscr{A}$ admits a left (resp., right) adjoint φ, which we call the *reflection* (resp., *coreflection*).

If every A in $\mathscr{A}$ has a largest $\mathscr{C}$ subobject, the choice of $A_{\mathscr{C}}$ for each A determines a right adjoint to the inclusion of $\mathscr{C}$ in $\mathscr{A}$, making $\mathscr{C}$ a coreflective subcategory of $\mathscr{A}$.

The assumption that $\mathscr{A}$ is cocomplete and well-powered will be crucial for the following result.

Proposition 2.1.2. *For a cocomplete well-powered abelian category $\mathscr{A}$ and any full subcategory $\mathscr{C}$ of $\mathscr{A}$, closed under sums and quotients in $\mathscr{A}$, any A in $\mathscr{A}$ has a largest $\mathscr{C}$-subobject.*

Proof. Let A be an object of $\mathscr{A}$, and let $\{C_i\}$ be the set of subobjects of A in $\mathscr{C}$. Write $A_{\mathscr{C}}$ for the image of $\oplus_i C_i$ in A. Since $\mathscr{C}$ is closed under sums and quotients, $A_{\mathscr{C}}$ is the desired maximal subobject of A in $\mathscr{C}$. □

We now define some key notions in torsion theory.

Definition 2.1.3. 1. A *quotient functor* is an endofunctor $\varphi : \mathscr{A} \longrightarrow \mathscr{A}$ together with a natural epimorphism $\eta : \mathrm{id} \longrightarrow \varphi$. That is, for every $f : A \longrightarrow B$, the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\ \varphi(A) & \xrightarrow{\varphi(f)} & \varphi(B) \end{array}$$

We will often drop the reference to η.

2. We say that φ is *idempotent* if the natural epimorphism is the identity on the essential image of φ. That is, $\eta_{\varphi(A)} : \varphi(A) \longrightarrow \varphi^2(A)$ is a natural isomorphism.

3. A quotient functor φ is a *pre-coradical* if for all A in $\mathscr{A}$, φ applied to the kernel of the epimorphism $A \longrightarrow \varphi(A)$ is 0.

4. Finally, a pre-coradical φ is a *coradical* if φ is right exact.

Remark 2.1.4. Notice that quotient functors always take epimorphisms to epimorphisms. However, pre-coradicals are not always right exact. If $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence, and φ is a pre-coradical, exactness can fail at $\varphi(A)$.

Definition 2.1.5. Let $\varphi : \mathscr{A} \longrightarrow \mathscr{A}$ be an endofunctor of an abelian category $\mathscr{C}$. We say that φ is a *pre-radical* if there exists a natural monomorphism $\varphi \longrightarrow \mathrm{id}$ (in which case, we say that φ is a *subobject functor*) such that $\varphi(A/\varphi(A)) = 0$ for all A. If φ is also left-exact, then φ is a *radical.*

Example 2.1.6. Let **Ab** be the category of abelian groups, and let G an abelian group, written additively. We write G_{tor} for the torsion subgroup of G, and we write $\varphi(G)$ for G/G_{tor}. The quotient functor φ is a pre-coradical, but is not a coradical. To see this, consider the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

in the category of **Ab**. Applying φ, we have

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

which is not exact in the middle. On the other hand, it is easy to see that the functor $G \mapsto G_{tor}$ is a radical.

More generally, let R be a commutative ring, and S be a multiplicatively closed set. For an R module M, let ${}_SM$ be the submodule M of elements annihilated by S. We write $\varphi(M)$ for $M/{}_SM$. Then φ defines a pre-coradical on the category of R-modules. As in the case for abelian groups, the functor φ is not a coradical.

Torsion theory is usually developed for radicals, which are coradicals in the opposite category of $\mathscr{A}$. However, throughout this chapter, we mostly consider statements for (pre-)coradicals. We leave the dual statements to the reader to formulate or look up in [Dic66] or [BJV, Section 1.2].

Proposition 2.1.7. *Any right exact quotient functor φ of an abelian category $\mathscr{A}$ is idempotent. In particular, any coradical is idempotent (* cf. *[BJV, I2.2]).*

Proof. Fix A in $\mathscr{A}$, and let η denote the natural epimorphism associated to the quotient functor φ. Let K be the kernel of $\eta_A : A \longrightarrow \varphi(A)$, and consider the sequence

$$0 \longrightarrow K \longrightarrow A \longrightarrow \varphi(A) \longrightarrow 0.$$

Applying φ, which is right exact, we have

$$\varphi(K) \longrightarrow \varphi(A) \longrightarrow \varphi^2(A) \longrightarrow 0.$$

Thus, $\varphi^2(A)$ is the cokernel of $\varphi(K) \longrightarrow \varphi(A)$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} K & \longrightarrow & A \\ \downarrow{\scriptstyle \eta_K} & & \downarrow{\scriptstyle \eta_A} \\ \varphi(K) & \longrightarrow & \varphi(A) \end{array}$$

and, since $K \longrightarrow \varphi(K)$ is an epimorphism,

$$\begin{aligned} \varphi^2(A) &= \text{cok } (\varphi(K) \longrightarrow \varphi(A)) \\ &= \text{cok } (K \longrightarrow \varphi(K) \longrightarrow \varphi(A)) \\ &= \text{cok } (K \longrightarrow A \longrightarrow \varphi(A)). \end{aligned}$$

But $K \longrightarrow A \longrightarrow \varphi(A)$ is the 0 map. Therefore, $\varphi^2(A) = \varphi(A)$ as desired. □

In addition to being dual notions, there is a one-to-one correspondence between idempotent pre-radicals and idempotent pre-coradicals:

Proposition 2.1.8. *Let φ be an idempotent pre-coradical of an abelian category $\mathscr{A}$, and η be its corresponding natural epimorphism. Write $\kappa(A)$ for* ker $(A \xrightarrow{\eta_A} \varphi(A))$. *Then κ is a pre-radical.*

Dually, if ψ is an idempotent pre-radical with natural injection ϵ. Writing $\gamma(A) =$ cok ϵ_A, *we have that γ is a pre-coradical.*

Proof. It suffices to prove this statement for the idempotent pre-coradicals, as the statement for pre-radical is the dual assertion. We proceed as follows:

The fact that κ is functorial follows from the naturality of η. Moreover, it is clear that κ is a subobject functor. To see that κ is also a pre-radical, we need to show that $\kappa(A/\kappa(A)) = 0$ for all A in $\mathscr{A}$. Fix such an A, and notice that $A/\kappa(A) = \varphi(A)$. Then we have the associated short exact sequence:

$$0 \longrightarrow \kappa(\varphi(A)) \longrightarrow \varphi(A) \longrightarrow \varphi^2(A) \longrightarrow 0.$$

But $\varphi(A) \longrightarrow \varphi^2(A)$ is the identity. It follows that $\kappa(\varphi(A)) = \kappa(A/\kappa(A)) = 0$.

Next, consider the following short exact sequence associated to $\kappa(A)$:

$$0 \longrightarrow \kappa(\kappa(A)) \longrightarrow \kappa(A) \overset{\eta_{\kappa(A)}}{\longrightarrow} \varphi(\kappa(A)) \longrightarrow 0.$$

Since φ is a pre-coradical, we have we have that

$$\varphi(\kappa(A)) = \varphi(\ker\,(A \longrightarrow \varphi(A))) = 0.$$

It follows that $\kappa^2(A) = \kappa(A)$, and κ is idempotent. The proposition follows. □

Proposition 2.1.9. *Let φ be a pre-coradical of an abelian category $\mathscr{A}$. Suppose B is a quotient of $\varphi(A)$, and let K be the kernel of the composition $A \longrightarrow \varphi(A) \longrightarrow B$. Then $\varphi(K)$ is isomorphic to the kernel of the epimorphism $\varphi(A) \longrightarrow B$ (*cf. *[BJV, I2.3]).*

Proof. Let η denote the natural epimorphism associated to the quotient functor φ, and let f denote the epimorphism given by the composition $A \longrightarrow \varphi(A) \longrightarrow B$.

Consider the exact sequence $0 \longrightarrow K \longrightarrow A \longrightarrow B \longrightarrow 0$. We claim that $\varphi(K) \longrightarrow \varphi(A) \longrightarrow B \longrightarrow 0$ is exact and fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{g} & A & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_K} & & \downarrow{\scriptstyle \eta_A} & & \| & & \\ & & \varphi(K) & \xrightarrow{\varphi(g)} & \varphi(A) & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

Notice that $\eta_K : K \longrightarrow \varphi(K)$ is epi. Therefore, the cokernel of $\varphi(g)$ is the cokernel of $K \longrightarrow A \longrightarrow \varphi(A)$. But the epimorphism $A \longrightarrow B$ factors through $\varphi(A)$. It follows that cok $\varphi(g) = B$. The rest of the claim now follows.

Let L be the kernel of η_A. We claim that L is also the kernel of the η_K. Since f factors through $A \longrightarrow \varphi(A)$, there exists a map from L to K, which we call h. Applying the Snake Lemma to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & A & \longrightarrow & \varphi(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \| & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

we have that L is a subobject of K. Let L' be the kernel of $K \longrightarrow \varphi(K)$. We claim that L is isomorphic to L'. Notice that we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \xrightarrow{i'} & K & \xrightarrow{\eta_K} & \varphi(K) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle j} & \nearrow{\scriptstyle h} & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle \varphi(g)} & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & A & \xrightarrow{\eta_A} & \varphi(A) & \longrightarrow & 0. \end{array} \tag{2.1.10}$$

Since $\eta_A \circ g \circ i' = \varphi(g) \circ \eta_K \circ i' = 0$, there exists a map j from L' to L (dotted arrow in (2.1.10)) such that $ij = gi'$. Applying the Snake Lemma to (2.1.10), we see that j is injective.

By the naturality of η, we also have the following commutative square:

$$\begin{array}{ccc} L & \xrightarrow{h} & K \\ \Big\downarrow{\scriptstyle \eta_L} & & \Big\downarrow{\scriptstyle \eta_K} \\ \varphi(L) & \longrightarrow & \varphi(K). \end{array}$$

Since φ is a pre-coradical, $\varphi(L) = 0$. Therefore, $\eta_K \circ h = 0$. Thus, there exists a map $j' : L \longrightarrow L'$ such that $j \circ j' = \mathrm{id}_L$ and $j' \circ j = \mathrm{id}_{L'}$. It follows that $L \cong L'$. Applying the Snake Lemma to (2.1.10), we see that $\varphi(K) \longrightarrow \varphi(A)$ is injective, as desired. $\square$

2.2 Torsion theories and coradicals

Definition 2.2.1. A *torsion theory* for an abelian category $\mathscr{A}$ is a pair $(\mathscr{T}, \mathscr{F})$ of full subcategories, called the *torsion subcategory* and the *torsion-free subcategory* respectively, where the objects of $\mathscr{T}$ are the objects T such that $\mathrm{Hom}_{\mathscr{A}}(T, F) = 0$ for every F in $\mathscr{F}$ and the objects of $\mathscr{F}$ are the objects F such that $\mathrm{Hom}_{\mathscr{A}}(T, F) = 0$ for every object T in $\mathscr{T}$.

Certainly $0 \in \mathscr{T} \cap \mathscr{F}$. Therefore, neither subcategory is empty. We also have the following characterization of the torsion and torsionfree subcategories.

Proposition 2.2.2. *Suppose $\mathscr{T}$ and $\mathscr{F}$ are two full subcategories of a cocomplete well-powered abelian category $\mathscr{A}$. Then $\mathscr{T}$ is the torsion subcategory of a torsion theory of $\mathscr{A}$ if and only if $\mathscr{T}$ is closed under extensions, direct sums and quotients.*

*Dually, $\mathscr{F}$ is a torsionfree subcategory of a torsion theory of $\mathscr{A}$ if and only if $\mathscr{F}$ is closed under extensions, direct products, and subobjects. (*cf. *[BJV, I2.6])*

Proof. It suffices to verify the statement for torsion subcategories. Suppose $\mathscr{T}$ is a torsion subcategory with $\mathscr{F}'$ its corresponding torsionfree subcategory.

Closed under quotients: suppose T is an object of $\mathscr{T}$. For any epimorphism $T \longrightarrow T'$, we have

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(T', F) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(T, F)$$

for any F in $\mathscr{F}'$. However, $\mathrm{Hom}_{\mathscr{A}}(T, F) = 0$. Therefore, $\mathrm{Hom}_{\mathscr{A}}(T', F) = 0$ for all F, and Y is in $\mathscr{T}$.

Closed under sums: suppose $\{T_i\}_{i\in I}$ is a collection of objects of $\mathscr{T}$. We have

$$\mathrm{Hom}_{\mathscr{A}}(\oplus_{i\in I} T_i, F) = \prod_{i\in I} \mathrm{Hom}_{\mathscr{A}}(T_i, F) = 0$$

for all F in $\mathscr{F}'$. It follows that $\oplus_{i\in I} T_i$ is an object of $\mathscr{T}$.

Closed under extensions: Suppose

$$0 \longrightarrow T' \longrightarrow A \longrightarrow T'' \longrightarrow 0$$

is an exact sequence in $\mathscr{A}$ with $T', T'' \in \mathscr{T}$. Then for any F in $\mathscr{F}$,

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(T'', F) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, F) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(T', F).$$

Since $\mathrm{Hom}_{\mathscr{A}}(T'', F) = \mathrm{Hom}_{\mathscr{A}}(T', F) = 0$, it follows that $\mathrm{Hom}_{\mathscr{A}}(A, F) = 0$ for all F. Therefore, A is in $\mathscr{T}$.

Conversely, suppose $\mathscr{T}$ is closed under extensions, direct sums and quotients. Let $\mathscr{F}'$ be the full subcategory of F such that $\mathrm{Hom}_{\mathscr{A}}(T, F) = 0$ for all T in $\mathscr{T}$, and let $\mathscr{T}'$ be the full subcategory of $\mathscr{A}$ whose objects are all T' such that $\mathrm{Hom}_{\mathscr{A}}(T', F) = 0$ for all F in $\mathscr{F}$. We claim that $\mathscr{T}' = \mathscr{T}$.

Clearly, $\mathscr{T}$ is a full subcategory of $\mathscr{T}'$. Let T be an object of $\mathscr{T}'$. By Proposition 2.1.2, there exists a maximal $\mathscr{T}$-subobject of T, which we represent by $T_{\mathscr{T}}$. We show that $T/T_{\mathscr{T}}$ is an object of $\mathscr{F}'$, and therefore it must be 0.

Suppose not. Then there exists some T' in $\mathscr{T}$ with a nonzero map $f : T' \longrightarrow T/T_{\mathscr{T}}$. Since $f(T')$ is an object in $\mathscr{T}$, replacing T' by its image in $T/T_{\mathscr{T}}$, we may assume without loss of generality that f is monic.

Pull back $T \longrightarrow T/T_{\mathscr{T}}$ by f, and we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\mathscr{T}} & \longrightarrow & P & \xrightarrow{p} & T' & \longrightarrow & 0 \\ & & \| & & \downarrow i & & \downarrow f & & \\ 0 & \longrightarrow & T_{\mathscr{T}} & \longrightarrow & T & \longrightarrow & T/T_{\mathscr{T}} & \longrightarrow & 0 \end{array}$$

As i is a pullback of a monomorphism, i is itself monic. As $T \longrightarrow T/T_{\mathscr{T}}$ is epimorphic, so is p. Furthermore, $\ker p = T_{\mathscr{T}}$. Since $T_{\mathscr{T}}$ and T' are both in $\mathscr{T}$, it follows that P must be in $\mathscr{T}$ as well. However, T' is nontrivial, contradicting the maximality of $T_{\mathscr{T}}$. Thus, $T/T_{\mathscr{T}} \in \mathscr{F}$, and $T \in \mathscr{T}$. □

Proposition 2.2.3. *Let $(\mathscr{T}, \mathscr{F})$ be a pair of full subcategories of a cocomplete well-powered abelian category $\mathscr{A}$. Then $(\mathscr{T}, \mathscr{F})$ is a torsion theory if and only if the following conditions hold:*

1. *the only common object of $\mathscr{T}$ and $\mathscr{F}$ is 0.*
2. *for every A in $\mathscr{A}$, there exists a subobject $A_{\mathscr{T}}$ of A in $\mathscr{T}$ such that $A/A_{\mathscr{T}}$ is an object of $\mathscr{F}$.*

*(*cf. *[BJV, I2.7])*

Proof. $\Rightarrow$: Suppose A is $\mathscr{T} \cap \mathscr{F}$. Then $\mathrm{Hom}_{\mathscr{A}}(A, A) = 0$, so the identity is the zero map, and $A = 0$. Now, for A in $\mathscr{A}$, let $A_{\mathscr{T}}$ be its maximal $\mathscr{T}$ subobject. By the same reasoning as in the previous proposition, $A/A_{\mathscr{T}}$ is an object of $\mathscr{F}$.

$\Leftarrow$: suppose $\mathscr{T}, \mathscr{F}$ satisfy the condition of the proposition, and there is some A in $\mathscr{A}$ such that for all F in $\mathscr{F}$, $\mathrm{Hom}_{\mathscr{A}}(A, F) = 0$. Let $A_{\mathscr{T}}$ denote the $\mathscr{T}$-subobject in Condition (2) associated to A. Since $A/A_{\mathscr{T}} \in \mathscr{F}$, $A \longrightarrow A/A_{\mathscr{T}}$ is the zero map. Hence, $A = A_{\mathscr{T}}$, and A is in $\mathscr{T}$. Similarly, if $F \in \mathscr{F}$, then the inclusion $F_{\mathscr{T}} \longrightarrow F$ is the zero map, and hence $F/F_{\mathscr{T}} = F$ which is in $\mathscr{F}$. □

Proposition 2.2.4. *Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory for a cocomplete well-powered abelian category $\mathscr{A}$. Sending A in $\mathscr{A}$ to its largest $\mathscr{T}$-subobject $A_{\mathscr{T}}$ defines an idempotent pre-radical.*

*Dually, sending A to $A/A_{\mathscr{T}}$ defines an idempotent pre-coradical (*cf. *[BJV, I2.8]).*

Proof. In this case, it is easier to prove the statement for idempotent pre-radicals. Let κ denote the association defined by $A \mapsto A_{\mathscr{A}}$ for A in $\mathscr{A}$.

To see that κ is a functor, let $f : A \longrightarrow B$ be any morphism. The image of $\kappa(A)$ in B under f is in $\mathscr{T}$. By the maximality of $\kappa(B)$, there exists a map $g : f(\kappa(A)) \longrightarrow \kappa(B)$, and define the map $\kappa(f)$ to be the composition of $gf|_{\kappa(A)}$.

It is clear from the construction that κ is a subobject functor. Since $\kappa(A) \in \mathscr{T}$, it is clear that the largest suboboject of $\kappa(A)$ is itself: hence $\varphi^2(A)) = \varphi(A)$. By the maximality of $\kappa(A)$, $A/\kappa(A) \in \mathscr{F}$, and

$$\kappa(A/\kappa(A)) = 0.$$

The dual statement follows from Proposition 2.1.8. □

Remark 2.2.5. Since a coradical φ is left adjoint to the inclusion of its associated torsionfree subcategory $\mathscr{F}$ in $\mathscr{A}$ and its associated idempotent pre-radical κ is right adjoint to the inclusion of the torsion subcategory $\mathscr{T}$ in $\mathscr{A}$, $\mathscr{T}$ is a coreflective subcategory, and $\mathscr{F}$ is a reflective subcategory of $\mathscr{A}$.

Theorem 2.2.6. *Let $\mathscr{A}$ be a cocomplete well-powered abelian category. There is a one-to-one correspondence between isomorphism classes of idempotent pre-coradicals of $\mathscr{A}$ and torsion theories for $\mathscr{A}$. If φ is a pre-coradical, and η is its associated natural epimorphism, then the torsion theories are defined by*

$$\mathscr{T} = \{T|\ \varphi(T) = 0\}$$

$$\mathscr{F} = \{F|\ \eta_F : F \longrightarrow \varphi(F) \text{ is an isomorphism}\}.$$

*(*cf. *[BJV, I2.9]).*

Proof. Obtaining an idempotent pre-coradical from a torsion theory is established by Proposition 2.2.4. Therefore, it suffices to show that $(\mathscr{T}, \mathscr{F})$ as given in the statement of the theorem defines a torsion theory on $\mathscr{A}$, and that the associations define quasi-inverses of one another.

To do this, we appeal to Proposition 2.2.3. It is clear that the only object common to both $\mathscr{T}$ and $\mathscr{F}$ is 0. So we need only to show that for every A in $\mathscr{A}$, there exists T in $\mathscr{T}$ such that $A/T \in \mathscr{F}$.

Fix A in $\mathscr{A}$. Since φ is idempotent, $\varphi(A)$ is in $\mathscr{F}$. Since φ is a pre-coradical, the kernel of $A \longrightarrow \varphi(A)$ is in $\mathscr{T}$. $\square$

As we have mentioned in the paragraph preceding Proposition 2.1.7, there is a result corresponding to Theorem 2.2.6 for radicals: the isomorphism classes of idempotent pre-radical κ are in one-to-one correspondence with torsion theories on $\mathscr{A}$. For a given pre-radical κ with natural inclusion ϵ, the associated torsion theory is defined by

$$\mathscr{T} = \{T | \epsilon_F : \kappa(T) \longrightarrow T \text{ is an isomorphism}\}$$
$$\mathscr{F} = \{F | \kappa(F) = 0\}.$$

In fact, we have the following.

Corollary 2.2.7. *Let φ be an idempotent pre-coradical, and let κ be the idempotent pre-radical associated to φ (see Proposition 2.1.8). Then the torsion theory defined by φ in Theorem 2.2.6 is the same as the one for κ as defined above.*

Moreover, φ is left adjoint to the inclusion $\mathscr{F} \longrightarrow \mathscr{A}$ and κ is right adjoint to the inclusion $\mathscr{T} \longrightarrow \mathscr{A}$.

Proof. The only thing left to verify is that φ defines a left adjoint to the inclusion of $\mathscr{F}$ into $\mathscr{A}$ and κ defines a right adjoint to the inclusion of $\mathscr{T}$ into $\mathscr{A}$. We verify the statement only for φ and leave the latter to the reader.

For φ, let A be an object of $\mathscr{A}$, and let F be an object of $\mathscr{F}$. Consider the short exact sequence

$$0 \longrightarrow \kappa(A) \longrightarrow A \longrightarrow \varphi(A) \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathscr{A}}(-, F)$, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(\varphi(A), F) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(A, F) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(\kappa(A), F)$$

Since $\kappa(A) \in \mathscr{T}$ (Theorem 2.2.6) and $\varphi(A) \in \mathscr{F}$, $\mathrm{Hom}_{\mathscr{A}}(\kappa(A), F) = 0$, and

$$\mathrm{Hom}_{\mathscr{A}}(\varphi(A), F) = \mathrm{Hom}_{\mathscr{F}}(\varphi(A), F) \cong \mathrm{Hom}_{\mathscr{A}}(A, F)$$

as desired. $\square$

It should be evident from Theorem 2.2.6 that isomorphism classes of coradicals are not in one-to-one correspondence with torsion theories, although they do give rise to unique torsion theories. We now characterize the properties of the torsion theories that arise from coradicals.

Definition 2.2.8. Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory on $\mathscr{A}$. We say that $(\mathscr{T}, \mathscr{F})$ is *hereditary* if $\mathscr{T}$ is closed with respect to subobjects. That is, if $A \hookrightarrow B$ is an monomorphism in $\mathscr{A}$ such that $B \in \mathscr{T}$, then $A \in \mathscr{T}$.

Dually, we say that $(\mathscr{T}, \mathscr{F})$ is *cohereditary* if $\mathscr{F}$ is closed under quotients.

Theorem 2.2.9. *Let $\mathscr{A}$ be a cocomplete well-powered abelian category. There is a one-to-one correspondence between isomorphism classes of coradicals of $\mathscr{A}$ and cohereditary torsion theories on $\mathscr{A}$ (*cf. *[BJV, I2.12]).*

Proof. From coradicals to cohereditary torsion theories: Let φ be a coradical with natural epimorphisms η, and $(\mathscr{T}, \mathscr{F})$ the associated torsion theory, given by Theorem 2.2.6.

We need only to show that $\mathscr{F}$ defined by $\{A|\eta : A \longrightarrow \varphi(A) \text{ is an isomorphism}\}$ is closed under quotients. Let $f : F \longrightarrow A$ be an epimorphism with F in $\mathscr{F}$, and write K for the kernel of f. It follows from Proposition 2.2.2 that $\mathscr{F}$ is closed under subobjects. Hence, $K \in \mathscr{F}$. Furthermore, by the right exactness of φ, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & F & \xrightarrow{f} & A & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow & & \\ & & \varphi(K) & \xrightarrow{\varphi(i)=i} & \varphi(F) & \xrightarrow{\varphi(f)} & \varphi(A) & \longrightarrow & 0, \end{array}$$

whence $A = \varphi(A)$ as desired. It follows that $A \in \mathscr{F}$.

From cohereditary torsion theory to coradicals: Let $(\mathscr{T}, \mathscr{F})$ be a cohereditary torsion theory on $\mathscr{A}$, and let φ be its associated idempotent pre-coradical given by Theorem 2.2.6. We need to show that φ is right exact.

We begin by demonstrating that, for an epimorphism $f : A \longrightarrow B$ in $\mathscr{A}$, $\varphi(B)$ is isomorphic to the push-out P of f and the natural epimorphism $\eta_A : A \longrightarrow \varphi(A)$, as in the following diagram:

$$\begin{CD} A @>>> B \\ @VVV @VVV \\ \varphi(A) @>>> P. \end{CD}$$

Since φ is left adjoint to inclusion, we have that

$$\mathrm{Hom}_{\mathscr{A}}(\varphi(B), F) = \mathrm{Hom}_{\mathscr{F}}(\varphi(B), F) = \mathrm{Hom}_{\mathscr{A}}(B, F)$$

for all F in $\mathscr{F}$. In particular, for all F in $\mathscr{F}$, and epimorphisms $B \twoheadrightarrow F$, there exists a unique map $\varphi(B) \longrightarrow F$ making the following diagram commutative

$$\begin{array}{ccc} B & \longrightarrow & F \\ & \searrow^{\eta_B} \quad \nearrow & \\ & \varphi(B). & \end{array}$$

Now, since P is the push-out, and $\varphi(B)$ fits into the following commutative diagram

$$\begin{CD} A @>>> B \\ @V{\eta_A}VV @VV{\eta_B}V \\ \varphi(A) @>>> \varphi(B) \end{CD} \tag{2.2.10}$$

there exists an unique map $P \longrightarrow \varphi(B)$. Furthermore, the map $\varphi(A) \longrightarrow P$ is an epimorphism since it is the push-out of the epimorphism $A \longrightarrow B$. Since $\varphi(A) \in \mathscr{F}$, which is closed under quotients, it follows that $P \in \mathscr{F}$. The map $B \longrightarrow P$ is also an epimorphism because it is the push-out of the epimorphism $A \longrightarrow \varphi(A)$. It follows by the previous point that there exists an unique map $\varphi(B) \longrightarrow P$.

Since both maps are unique, it follows that each map is an isomorphism and is the inverse of the other. Furthermore, Diagram (2.2.10) is a push-out diagram.

To complete the proof that φ is right exact, we need only to show that for an exact sequence

$$0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$$

in $\mathscr{A}$, $\varphi(A'')$ is the cokernel of $\varphi(A') \longrightarrow \varphi(A)$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{f} & A & \xrightarrow{g} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_{A'}} & & \downarrow{\scriptstyle \eta_{A}} & & \downarrow{\scriptstyle \eta_{A''}} & & \\
 & & \varphi(A') & \xrightarrow{\varphi(f)} & \varphi(A) & \xrightarrow{\varphi(g)} & \varphi(A'') & \longrightarrow & 0 \\
 & & & & & \searrow{\scriptstyle h} & \uparrow{\scriptstyle p} & & \\
 & & & & & & \operatorname{cok} \varphi(f) & \longrightarrow & 0,
\end{array}$$

where the top row is exact. Since the composition $\varphi(A') \longrightarrow \varphi(A) \longrightarrow \varphi(A'')$ is 0, there exists an unique map $\operatorname{cok} \varphi(f) \xrightarrow{p} \varphi(A'')$ (shown as the dotted arrow in the diagram above), such that $ph = \varphi(g)$.

However, we also have that $h \circ \eta_A \circ f = h \circ \varphi(f) \circ \eta_{A'} = 0$. It follows that there exists a map $A'' \longrightarrow \operatorname{cok} \varphi(f)$ (represented by the dotted arrow in the following diagram) such that the following diagram is commutative:

$$\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{\scriptstyle \eta_Y} & & \downarrow \\
\varphi(Y) & \longrightarrow & \operatorname{cok} \varphi(f).
\end{array}$$

But $\varphi(A'')$, as a push-out, admits an unique map $\varphi(A'') \xrightarrow{p'} \operatorname{cok} \varphi(f)$. Once again, since the maps defined between $\varphi(A'')$ and $\operatorname{cok} \varphi(f)$ are unique with respect to $\varphi(g)$ and h, it follows that p and p' are isomorphisms and define inverses of one another. This concludes the theorem. □

Remark 2.2.11. Notice that if φ is a coradical, then $\mathscr{F}$ is a *Serre subcategory* of $\mathscr{A}$. In particular, $\mathscr{F}$ is an abelian category. In the case when $\mathscr{A}$ has "enough $\mathscr{F}$-covers", then the torsion subcategory $\mathscr{T}$ is precisely the localization of $\mathscr{A}$ by $\mathscr{F}$ in the sense of [Swan], and the associated idempotent radical κ is an exact radical.

Chapter 3

Homotopy Invariant Sheaves with Transfers

In this chapter, we define the notion of homotopy invariant Nisnevich sheaves with transfers. In order to do so, we need to introduce the category of correspondences and presheaves with transfers.

For the remainder of the thesis, let k be a perfect field, and let Sm_k denote the category of smooth separated finite type k-schemes. The material in this chapter is taken from Lecture 2 and 6 of [MVW].

3.1 Sheaves with Transfers

Definition 3.1.1. Let X, Y be smooth separated k-schemes. An *elementary correspondence from X to Y* is an irreducible closed subset W of $X \times Y$ such that the projection to X from the associated integral subscheme $\overline{W}$ is finite and surjective onto a component of X.

Let $Cor_k(X, Y)$ (or simply $Cor(X, Y)$ in the case when the base field k is understood) denote the free abelian group generated by the elementary correspondences from X to Y. Elements of $Cor(X, Y)$ are called *finite correspondences from X to Y*.

Example 3.1.2. In the case when X is an integral scheme over k, the graph of any morphism $\varphi : X \longrightarrow Y$ defines an elementary correspondence from X to Y.

In the case where $X = Y = \operatorname{Spec} L$, where L/k is a Galois extension, the elementary correspondences are precisely the graphs of the automorphisms in the Galois group $G \stackrel{\text{def}}{=} Gal(L, k)$. In this case, $Cor_k(X, Y) = \mathbb{Z}[G]$.

Let Cor_k be the collection of objects and morphisms where the objects of Cor_k are smooth separated finite type k-schemes and whose morphisms from X to Y in Cor_k

are given by $Cor(X,Y)$. We claim that Cor_k forms a category. The main missing piece here is a description of the composition of morphisms. We can define the composition of a finite correspondence V in $Cor(X,Y)$ with a finite correspondence W in $Cor(Y,Z)$ via the following construction taken from the discussion preceding [MVW, 1.5] and [MVW, 1.7]. Reduce to the case where X and Y are connected, and suppose that V and W are irreducible closed subsets of $X \times Y$ and $Y \times Z$ respectively. Let $\tilde{V}$ and $\tilde{W}$ be the underlying integral schemes associated to V and W. Then $\tilde{V} \times X$ and $X \times \tilde{W}$ define cycles in $X \times Y \times Z$ intersecting properly in the sense of [Ful84, 2.4]. Let T be the image of $\tilde{V} \times_Y \tilde{W}$ in $X \times Y \times Z$. Each irreducible component T_i of T is finite and surjective over X by [MVW, 1.7]. Furthermore, the image of T_i along $p : X \times Y \times Z \longrightarrow X \times Z$ is an irreducible closed subscheme of $X \times Z$ by [MVW, 1.4]. Let $[T]$ be the cycle corresponding to T in $X \times Y \times Z$. The push-forward $p_*([T])$ defines a finite correspondence from X to Z, which we define to be the composition $V \circ W$.

Definition 3.1.3. A *presheaf with transfers* is a contravariant functor from Cor_k to abelian groups (or R-modules for some commutative ring R). A map between presheaves F and G is a natural transformation from F to G. Let $\mathbf{PST}_k$ (or simply **PST** in the case when there is no ambiguity about the basefield k) denote the category of presheaves with transfers. Notice that **PST** has a natural structure of an abelian category.

Remark 3.1.4. The term "with transfers" comes from the existence of *transfer maps*. For F in **PST**, and a finite surjective morphism $\varphi : W \longrightarrow X$ of smooth schemes, there exists a map $\varphi_* : F(W) \longrightarrow F(X)$ induced by the graph of φ, regarded as an elementary correspondence from W to X. We call φ_* the *transfer map*. Notice that φ_* is in the "opposite direction" as the induced maps between sections.

Definition 3.1.5. ([SGA4, II.1.3]) A *Grothendieck pre-topology* on a category $\mathscr{C}$ is a collection $\mathscr{U}$ of *covering families* indexed by the objects of $\mathscr{C}$. Here, for each X in $\mathscr{C}$, a covering family of X is a collection of sets of morphisms $\{U_\alpha \longrightarrow X\}_\alpha$ called *covers of X*. Together, the covering families satisfy the following axioms:

1. for every map $Y \longrightarrow X$ in $\mathscr{C}$ and every cover $\{U_\alpha \longrightarrow X\}$ of X, the pullback

$Y \times U_\alpha \longrightarrow Y$ exists for every α, and $\{U_\alpha \times_X Y \longrightarrow Y\}$ is a cover of Y.

2. If $\{U_\alpha \longrightarrow X\}$ is a cover of X and for each α, $\{V_{\alpha\beta} \longrightarrow U_\alpha\}$ is a cover of U_α, then $\{V_{\alpha\beta} \longrightarrow X\}$ obtained via composition is a cover of X.

3. If $X' \longrightarrow X$ is an isomorphism, then $\{X' \longrightarrow X\}$ is a cover of X.

Remark 3.1.6. The notion of Grothendieck pre-topology generalizes the notion of a topology on a space X. Specifically, regarding a topology of X as a category $\mathscr{T}_X$ where the objects are open subsets of X and the morphisms are inclusion maps, then the collections $\{V_i \subset V\}$ of all covers of V, as V ranges over all open subsets of X satisfy the axioms of Definition 3.1.5 and define a Grothendieck pre-topology on $\mathscr{T}_X$.

Definition 3.1.7. For let $S \overset{\text{def}}{=} \{\varphi_\alpha : U_\alpha \longrightarrow X\}$ be a collection of morphisms between schemes. We say that S is *jointly surjective* if $\bigcup_{\varphi_\alpha \in S} \varphi_\alpha(U_\alpha) = X$.

Remark 3.1.8. For each X, consider the collection $\mathscr{U}_X$ of jointly surjective sets of open immersions $\{U_\alpha \longrightarrow X\}$. Then $\mathscr{U}_X$ as X ranges over all finite type k-schemes form a Grothendieck pre-topology $\mathscr{U}$ on the category Sch_k of finite type k-schemes called the *large Zariski site on k-schemes.*

We are interested in two other important Grothendieck pre-topologies on Sm_k. They are the étale site and the Nisnevich site, which we define below. Recall that a morphism $\varphi : X \longrightarrow Y$ is *étale* if φ is a flat and unramified. (See [Milne, §1.3].)

Definition 3.1.9. The *large étale site* on Sm_k, is the Grothendieck pre-topology given by a jointly surjective sets of étale morphisms $\{U_\alpha \longrightarrow X\}$.

The *large Nisnevich site* on Sm_k is the Grothendieck pre-topology such that every cover of X is an étale cover $\{U_\alpha \longrightarrow X\}$ such that for every x in X, there exists some $\varphi_\alpha : U_\alpha \longrightarrow X$ and y in U_α such that $\varphi_\alpha(y) = x$ and the induced map $k(x) \longrightarrow k(y)$ is an isomorphism.

Let $\mathrm{Sm}_{k,\text{ét}}$ and $\mathrm{Sm}_{k,\text{Nis}}$ denote respectively the étale and Nisnevich site of smooth schemes over k.

Since open immersions are étale, a jointly surjective collection of open immersions is both an étale cover and a Nisnevich cover. In this sense, the Zariski topology is coarser than the Nisnevich topology, which, in turn, is coarser than the étale topology on Sm_k.

Definition 3.1.10. An *étale sheaf with transfers* (resp. *Nisnevich sheaf with transfers*) F is a presheaf with transfers that is also an étale (resp. Nisnevich) sheaf. That is, F satisfies the following coherence conditions:

1. for each étale (resp. Nisnevich) cover $\{U_\alpha \longrightarrow X\}$, the following sequence is exact:
$$0 \longrightarrow F(X) \longrightarrow \prod_\alpha F(U_\alpha) \longrightarrow \prod_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$$
where the map $\prod_\alpha F(U_\alpha) \longrightarrow \prod_{\alpha,\beta} F(U_\alpha \times_X U_\beta)$ is given by the first and second projections from $U_\alpha \times_X U_\beta$ to U_α and U_β respectively for each α, β.

2. for each U, V, $F(U \sqcup V) = F(U) \oplus F(V)$.

We write $Sh_{\text{ét}}Cor$ (resp. $Sh_{\text{Nis}}Cor$) for the subcategory of étale (resp. Nisnevich) sheaves with transfers.

Since the category of sheaves on any locale is well-powered (see [Bo, 2.3.7]), the category of étale sheaves with transfers is also well-powered. So is the category of Nisnevich sheaves with transfers.

It is clear from the definition and the discussion following Definition 3.1.9 that an étale sheaf is also a Nisnevich sheaf, and an Nisnevich sheaf is a Zariski sheaf.

Our focus will be on Nisnevich sheaves with transfers, and here are some prominent examples.

Example 3.1.11. The constant sheaf $\mathbb{Z}$, the structure sheaf $\mathcal{O}$, and the sheaf of global units $\mathcal{O}^*$ are examples of étale and Nisnevich sheaves with transfers as defined in Definition 3.1.10. To see that $\mathbb{Z}, \mathcal{O}$, and $\mathcal{O}^*$ are étale and Nisnevich sheaves with transfers, we need to define the map $\varphi^* : \mathbb{Z}(Y) \longrightarrow \mathbb{Z}(X)$ (resp., $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$, $\mathcal{O}^*(Y) \longrightarrow \mathcal{O}^*(X)$) for every finite correspondence in $Cor(X, Y)$.

Assume that X and Y are integral schemes in Sm_k, and W is an elementary correspondence from X to Y. Then, W is given by an integral scheme finite over X, which we also represent by W. Let F and L be the function fields of X and W respectively. Then, L is an n-dimensional F-vector space, for some positive integer n. The induced map $\mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)$ is given by

$$\mathbb{Z} = \mathbb{Z}(Y) \xrightarrow{n} \mathbb{Z}(X) = \mathbb{Z}.$$

For the others, let $tr : L \longrightarrow F$ and $N : L \longrightarrow F$ denote the trace and norm maps respectively. Since X is normal and W is finite over X, tr and N restrict to homomorphisms $\mathcal{O}(W) \longrightarrow \mathcal{O}(X)$ and $\mathcal{O}^*(W) \longrightarrow \mathcal{O}^*(X)$ respectively. Hence, the map $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ is given by the composition

$$\mathcal{O}(Y) \longrightarrow \mathcal{O}(W) \xrightarrow{tr} \mathcal{O}(X),$$

and the map $\mathcal{O}^*(Y) \longrightarrow \mathcal{O}^*(X)$ is given by

$$\mathcal{O}^*(Y) \longrightarrow \mathcal{O}^*(W) \xrightarrow{N} \mathcal{O}^*(X).$$

Example 3.1.12. A large class of Nisnevich sheaves with transfers are the representable sheaves. For each X in Sm_k, write $\mathbb{Z}_{\mathrm{tr}}(X)$ for the sheaf which associates to each U the abelian group $Cor(U, X)$. To see that $\mathbb{Z}_{\mathrm{tr}}(X)$ is a Nisnevich sheaf, it suffices to show that it is an étale sheaf. In particular, for each X in Sm_k, $\mathbb{Z}_{\mathrm{tr}}(X)$ satisfies the coherence conditions given in Definition 3.1.10. The statement that $\mathbb{Z}_{\mathrm{tr}}(X)$ is an étale sheaf is proven in [MVW, 6.2].

Let $a_{\text{ét}}$ (resp. a_{Nis}) denote the étale (resp. Nisnevich) sheafification of a (general) presheaf on Sm_k. (See [Tamme, 3.1.1].) Furthermore, for a presheaf with transfers F, let $F_{\text{ét}}$ (resp. F_{Nis}) denote the étale (resp. Nisnevich) sheafification of F.

Proposition 3.1.13. *1. For F a presheaf with transfers, $F_{\text{ét}}$ has a unique structure of presheaf with transfers, and the canonical map $F \longrightarrow ({}_{\text{ét}}F)$ is a morphism of presheaves with transfers.*

The functor $a_{\text{ét}}$ restricted to **PST** *defines a left adjoint to the inclusion of $Sh_{\text{ét}}Cor$ into* **PST**.

Likewise, for F in **PST**, *F_{Nis} is a Nisnevich sheaf with transfers, and a_{Nis} restricted to* **PST** *defines a left adjoint to the inclusion of $Sh_{\mathrm{Nis}}Cor$ into* **PST**.

2. *Both $Sh_{\text{ét}}Cor$ and $Sh_{\text{Nis}}Cor$ are abelian subcategories of* $\mathbf{PST}$ *with enough injectives.*

Proof. For the statements about étale sheaves with transfers, see [MVW, 6.17, 6.18 and 6.19]. The arguments in the proofs of the statements about étale sheaves can easily be extended to proofs for the Nisnevich sheaves. □

3.2 Homotopy invariant sheaves with transfers

We now introduce the notion of homotopy invariant sheaves (defined below), which will play a central role in the subsequent chapters.

Definition 3.2.1. A presheaf F is *homotopy invariant* if the map

$$F(X) \longrightarrow F(X \times \mathbb{A}^1)$$

induced by the projection $X \times \mathbb{A}^1 \longrightarrow X$ is an isomorphism. We write $\mathbf{HI}_{pre}$ for the category of homotopy invariant presheaves with transfers.

Similarly, we define homotopy invariant sheaves, and write $\mathbf{HI}_{\text{ét}}$ (resp. $\mathbf{HI}_{\text{Nis}}$) for the full subcategory of homotopy invariant étale sheaves (resp. Nisnevich sheaves) with transfers. We will simply write $\mathbf{HI}$ when the underlying pre-topology is understood.

Since $Sh_{\text{ét}}Cor$ and $Sh_{\text{Nis}}Cor$ are both well-powered, so are $\mathbf{HI}_{\text{ét}}$ and $\mathbf{HI}_{\text{Nis}}$.

Remark 3.2.2. If F is a homotopy invariant presheaf with transfers, then $F_{\text{ét}}$ and F_{Nis} are homotopy invariant sheaves under the étale and Nisnevich topologies respectively. Together with Proposition 3.1.13, we have the following commutative diagram detailing the subcategories of presheaves on Sm_k and their reflection functors:

$$\begin{array}{ccccc}
\mathbf{HI}_{\text{ét}} & \longrightarrow & Sh_{\text{ét}}Cor & \longrightarrow & Sh_{\text{ét}} \\
{\scriptstyle a_{\text{ét}}}\uparrow\downarrow & & {\scriptstyle a_{\text{ét}}}\uparrow\downarrow & & {\scriptstyle a_{\text{ét}}}\uparrow\downarrow \\
\mathbf{HI}_{\text{Nis}} & \longrightarrow & Sh_{\text{Nis}}Cor & \longrightarrow & Sh_{\text{Nis}} \\
{\scriptstyle a_{\text{Nis}}}\uparrow\downarrow & & {\scriptstyle a_{\text{Nis}}}\uparrow\downarrow & & {\scriptstyle a_{\text{Nis}}}\uparrow\downarrow \\
\mathbf{HI}_{pre} & \longrightarrow & \mathbf{PST} & \longrightarrow & PSh
\end{array}$$

where PSh denotes the presheaves on Sm_k. In the diagram above, the horizontal arrows represent forgetful functors, and the reflection functors in the first two columns are the restrictions of the reflection functors in the right-most column.

Of the three sheaves mentioned in Example 3.1.11, $\mathbb{Z}$ and $\mathcal{O}^*$ are homotopy invariant sheaves, and $\mathcal{O}$ is not. In fact, we can define a large class of homotopy invariant presheaves with transfers with the following construction:

Construction 3.2.3. Let F be a presheaf with transfers. Let Δ^n denote

$$\operatorname{Spec} k[x_0, \ldots, x_n]/\big(1 - \sum_i x_i\big).$$

Notice that for each i in $\{0, ..., n\}$, there exists a map $\partial_{n,i} : \Delta^{n-1} \longrightarrow \Delta^n$ induced by

$$k[x_0, \ldots, x_n]/\big(\sum_i x_i - 1\big) \longrightarrow k[x_0, \ldots, x_{n-1}]/\big(\sum_i x_i - 1\big)$$

given by

$$x_j \mapsto \begin{cases} x_j & \text{if } j < i \\ 0 & \text{if } j = i \\ x_{j-1} & \text{otherwise.} \end{cases}$$

In particular, $\Delta^\bullet$ is a cosimplicial scheme, and $F(- \times \Delta^\bullet)$ is a simplicial presheaf with transfers. Let C_*F be the associated cochain complex. That is $(C_*F(X))^{-n} \stackrel{\text{def}}{=} C_nF(X) = F(X \times \Delta^n)$, and the chain map is given by

$$\partial_n^* \stackrel{\text{def}}{=} \sum_{i=0}^{n} (-1)^i \partial_{n,i}^*.$$

Clearly, if F is a homotopy invariant presheaf, then the complex C_*F is exact except at degree 0. In particular, the inclusion of F as a cochain complex concentrated in degree 0 into C_*F is a quasi-isomorphism of cochain complexes of presheaves. In general, for F in **PST**, write H^nC_*F for the contravariant functor $U \mapsto H^nC_*F(U)$. Then H^nC_*F is homotopy invariant for all n ([MVW, 2.19]).

If F is a sheaf with transfers, then C_nF is also a sheaf with transfers for all positive n. Therefore, C_*F is a cochain complex of sheaves with transfers. In particular, for all X in Sm_k, $C_*\mathbb{Z}_{\mathrm{tr}}(X)$ is a cochain complex of sheaves.

Definition 3.2.4. We write $h_X^{\text{ét}}$ (resp., h_X^{Nis}) for the etale (resp., Nisnevich) sheaf associated to $H^0C_*\mathbb{Z}_{\text{tr}}(X)$. In the case where the pre-topology is understood, we will omit the superscript, and simply write h_X for the associated sheaf.

Remark 3.2.5. Recall that two morphisms $f, g : X \longrightarrow Y$ in Cor are $\mathbb{A}^1$-*homotopic* if there exists some h in $Cor(X \times \mathbb{A}^1, Y)$ such that $h|_{X\times 0} = f$ and $h|_{X\times 1} = g$. We say that $f : X \longrightarrow Y$ is an $\mathbb{A}^1$-*homotopy equivalence* if there exists a $g : Y \longrightarrow X$ so that fg is homotopic to the identity on Y, and gf is homotopic to the identity on X.

If X and Y are homotopy equivalent, it is not true in general that $\mathbb{Z}_{\text{tr}}(X)$ is isomorphic to $\mathbb{Z}_{\text{tr}}(Y)$. For example, $\mathbb{Z}$ is obviously not isomorphic to $\mathbb{Z}_{\text{tr}}(\mathbb{A}^1)$. However, $C_*\mathbb{Z}_{\text{tr}}(X)$ is quasi-isomorphic to $C_*\mathbb{Z}_{\text{tr}}(Y)$ (see [MVW, 2.26]). Therefore, h_X and h_Y are isomorphic sheaves with transfers.

Remark 3.2.6. We note that all results of this chapter hold for $\mathbf{PST}(R)$, which are presheaves with transfers with values in R-modules, where R is some commutative unital ring. In particular, $R_{\text{tr}}(X)$ is an étale/Nisnevich sheaf, for every X in Sm_k.

We conclude this chapter with an endofunctor on the category **HI** that will play an important role in the construction of filtrations on **HI**.

Definition 3.2.7. Let F be a homotopy invariant presheaf with transfers. Write $F_{-1}(X)$ for the cokernel of $F(X \times \mathbb{A}^1) \longrightarrow F(X \times (\mathbb{A}^1 - 0))$. If F is a Nisnevich sheaf with transfers, then F_{-1} is again a Nisnevich sheaf with transfers by [MVW, 23.5]. We call F_{-1} the *the contraction of* F. We will write F_{-n+1} for $(F_{-n})_{-1}$.

If F is homotopy invariant, then F_{-1} is also a homotopy invariant. Furthermore, $F(X \times (\mathbb{A}^1 - 0))$ splits into $F(X) \oplus F_{-1}(X)$. Thus, if F is a sheaf, then F_{-1} is also a sheaf. In fact, $F \mapsto F_{-1}$ defines an endofunctor on the category of homotopy invariant sheaves with transfers.

Proposition 3.2.8 ([Dég08] 3.4.3)**.** *The functor* $F \mapsto F_{-1}$ *is exact.*

Chapter 4

The Derived Category of Motives

In this chapter, we define the derived category of motives $\mathbf{DM}^{\text{eff},-}$, and show that it is equipped with an additive symmetric monoidal structure with a partial internal hom (defined in Definition 4.2.1) that will be used to construct the slice filtration in Section 5 (see Remark 4.2.11).

To do this, we first define the bounded above derived category $\mathbf{D}^{-}ShCor$ of Nisnevich sheaves, and define $\mathbf{DM}^{\text{eff},-}$ to be the localization of $\mathbf{D}^{-}ShCor$ by a class of morphisms in $\mathbf{D}^{-}ShCor$ called $\mathbb{A}^1$-weak equivalences. We then show that $\mathbf{DM}^{\text{eff},-}$ is in fact equivalent as a category to the subcategory of $\mathbf{D}^{-}ShCor$ with homotopy invariant cohomology.

We also show that $\mathbf{D}^{-}\mathbf{PST}$ is equipped with tensor and internal hom operations on which induce a symmetric monoidal structure on $\mathbf{DM}^{\text{eff},-}$. For the remainder of the chapter, unless stated otherwise, all sheaves are Nisnevich sheaves. We will drop the "Nis", and simply write *ShCor* for the category of Nisnevich sheaves with transfers. This chapter is taken from Lectures 8, 9 and 14 of [MVW].

4.1 Derived Category of Motives

First consider the category **PST**. By Yoneda, for X in Sm_k and F in **PST**,

$$\mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), F) = F(X).$$

It follows that $\mathbb{Z}_{\mathrm{tr}}(X)$ is projective for every X in Sm_k. Since direct sums of projectives are projective, $\oplus_i \mathbb{Z}_{\mathrm{tr}}(X_i)$ is also projective for any arbitrary collection $\{Z_i\}$. Furthermore, for F in **PST**, there exists a surjection

$$\bigoplus_{X} \bigoplus_{x \in F(X)} \mathbb{Z}_{\mathrm{tr}}(X) \xrightarrow{x} F. \tag{4.1.1}$$

Hence, the category **PST** has enough projectives. Thus, we may define the bounded above derived category $\mathbf{D}^{-}\mathbf{PST}$ of the abelian category **PST** as the homotopy category of cochain complexes of projective objects in **PST** that are bounded above (see [Wei94, 10.4.8]). To construct the bounded above category of Nisnevich sheaves with transfers, we first need the following notion of a thick subcategory:

Definition 4.1.2. A full additive subcategory $\mathscr{W}$ of a derived category $\mathbf{D}$ is *thick* if it satisfies the following conditions:

1. if $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is a distinguished triangle, then any two of A, B, C is in $\mathscr{W}$, then so is the third.

2. if $A \oplus B$ is an object of $\mathscr{W}$, then A and B are both objects of $\mathscr{W}$.

If $\mathscr{W}$ is a thick subcategory of a derived category $\mathbf{D}$, then we can define a quotient triangulated category $\mathbf{D}/\mathscr{W}$. Let $\mathscr{S}$ be the set of maps whose cone is in $\mathscr{W}$. Then $\mathscr{S}$ is a saturated multiplicative system in the sense that $\mathscr{S}$ contains the identity, is closed under composition, and if $fg \in \mathscr{S}$, then f and g are both in $\mathscr{S}$. Define $\mathbf{D}/\mathscr{W}$ to be the localization $\mathbf{D}[\mathscr{S}^{-1}]$ (see [Verd96]).

Let $\mathscr{W}_{\mathrm{Nis}}$ be the system of morphisms between cochain complexes in $\mathbf{D}^{-}\mathbf{PST}$ inducing quasi-isomorphisms on the associated complex of Nisnevich sheaves. Since $\mathscr{W}_{\mathrm{Nis}}$ are the morphisms whose cone is in a thick subcategory, $\mathscr{W}_{\mathrm{Nis}}$ is saturated multiplicative system. We will write $\mathbf{D}^{-}ShCor$, or more simply $\mathbf{D}^{-}\mathbf{ST}$, for the bounded above derived category of Nisnevich sheaves with transfers, which is equivalent to the category obtained from $\mathbf{D}^{-}\mathbf{PST}$ by localizing with respect to $\mathscr{W}_{\mathrm{Nis}}$. We now define $\mathbf{DM}^{\mathrm{eff},-}$, the derived category of effective motives.

Definition 4.1.3. Let $\mathscr{W}_{\mathbb{A}}$ be the thick subcategory of $\mathbf{D}^{-}\mathbf{ST}$ generated by the cones of $\mathbb{Z}_{\mathrm{tr}}(X \times \mathbb{A}^1) \longrightarrow \mathbb{Z}_{\mathrm{tr}}(X)$ for every X in Sm_k, and closed under direct sums that exist in $\mathbf{D}^{-}\mathbf{ST}$. Write $\mathscr{S}_{\mathbb{A}}$ for the maps whose cone is in $\mathscr{W}_{\mathbb{A}}$. We say that a map f in $\mathbf{D}^{-}\mathbf{ST}$ is an $\mathbb{A}^1$-*weak equivalence* if $f \in \mathscr{S}_{\mathbb{A}}$.

We write $\mathbf{DM}^{\mathrm{eff},-}$ for the localization $\mathbf{D}^{-}\mathbf{ST}[\mathscr{S}_{\mathbb{A}}^{-1}]$. The category that we have just defined is the derived category of effective motives, whose objects are called *motives*.

While we have defined $\mathbf{DM}^{\text{eff},-}$ as a localization of $\mathbf{D}^-$ by the $\mathbb{A}^1$-weak equivalences, we can identify $\mathbf{DM}^{\text{eff},-}$ with a subcategory of $\mathbf{D}^-\mathbf{ST}$.

Definition 4.1.4. Let F^* be a cochain regarded as an object of $\mathbf{D}^-\mathbf{ST}$. We say that F^* is $\mathbb{A}^1$-*local* if $\operatorname{Hom}_{\mathbf{D}^-\mathbf{ST}}(-, F^*)$ sends $\mathbb{A}^1$-weak equivalences to isomorphisms. We write $\mathscr{L}$ for the full subcategory of $\mathbb{A}^1$-local objects in $\mathbf{D}^-\mathbf{ST}$.

Proposition 4.1.5 gives a good characterization of the category $\mathscr{L}$.

Proposition 4.1.5 ([MVW] Prop. 14.8, Cor. 14.9)**.** *For F^* in $\mathbf{D}^-\mathbf{ST}$, $F^* \in \mathscr{L}$ if and only if $a_{\text{Nis}}(H^n F^*)$ is homotopy invariant for every integer n. In particular, we can identify $\mathscr{L}$ with the full subcategory of complexes in $\mathbf{D}^-\mathbf{ST}$ with homotopy invariant cohomology presheaves.*

Definition 4.1.6. For F^* a bounded above cochain complex of sheaves with transfers, let CF^* denote the direct sum total complex of the double complex C_*F^*. Here, the (p, q) spot of the double complex C_*F^* is $C_{-p}F^q$. Therefore, CF^* again is an object of $\mathbf{D}^-\mathbf{ST}$.

Since F^* is bounded above, by shifting sufficiently, we may assume that $F^n = 0$ for $n > 0$. Therefore, indexing the double complex cohomologically, C_*F^* is a third quadrant double complex. Filtering the double complex C_pF^q by the second index q, we obtain a third quadrant spectral sequence converging to the cohomology of CF^*:

$$E_1^{p,q} = H^p(C_*F^q) \Rightarrow H^{p+q}(CF^*).$$

Since the cohomology presheaves $H^p(C_*F^q)$ are homotopy invariant for all p and q (see [MVW, 2.19]), the terms in the first page of the spectral sequence are all homotopy invariant. It follows that CF^* is in $\mathscr{L}$. The following proposition relates the construction defined above and the category $\mathscr{L}$.

Proposition 4.1.7. *The functor $C_* : \mathbf{D}^-\mathbf{ST} \longrightarrow \mathscr{L}$ is a left adjoint to the inclusion of $\mathscr{L} \hookrightarrow \mathbf{D}^-\mathbf{ST}$.*

Proof. There is a canonical map from $F^* \longrightarrow CF^*$, given by the inclusion of $F_i = C_0F_i \hookrightarrow \bigoplus_{p+q=i} C_{-p}F^q$. This map is a $\mathbb{A}^1$-weak equivalence (see [MVW, 14.4]).

Therefore, for any L^* in $\mathscr{L}$ and F^* in $\mathbf{D^-ST}$,

$$\mathrm{Hom}_{\mathbf{D^-ST}}(F^*, L^*) \cong \mathrm{Hom}_{\mathbf{D^-ST}}(CF^*, L^*) = \mathrm{Hom}_{\mathscr{L}}(CF^*, L^*). \qquad \square$$

There is a canonical functor $\pi : \mathbf{D^-ST} \longrightarrow \mathbf{DM}^{\mathrm{eff},-}$, given by sending an object of $\mathbf{D^-ST}$ to its corresponding object in $\mathbf{DM}^{\mathrm{eff},-}$. Its restriction to $\mathscr{L}$ defines a functor from $\mathscr{L}$ to $\mathbf{DM}^{\mathrm{eff},-}$.

Furthermore, we can define a map from $\mathbf{DM}^{\mathrm{eff},-}$ to $\mathscr{L}$. Notice that if F^* and F'^* are $\mathbb{A}^1$-weak equivalent, then transitivity implies that CF^* is $\mathbb{A}^1$-weak equivalent to CF^*. It follows that the functor that sends F^* to CF^* lifts to a functor from $\mathbf{DM}^{\mathrm{eff},-} \longrightarrow \mathscr{L}$. Let C_* denote the induced functor on $\mathbf{DM}^{\mathrm{eff},-}$.

Theorem 4.1.8. *The functor $\pi : \mathscr{L} \longrightarrow \mathbf{DM}^{\mathrm{eff},-}$ is an equivalence of categories, with quasi-inverse C_*.*

Proof. The fact that π is an equivalence is established in [MVW, 14.11]. Furthermore, given M in $\mathbf{DM}^{\mathrm{eff},-}$, then M is represented by some bounded above complex F^*. In turn, F^* is isomorphic (in $\mathbf{DM}^{\mathrm{eff},-}$) to CF^*, which is in the essential image of π, and define $C_*M = CF^*$.

For the second statement, it suffices at this point to show that $C_*\pi$ is naturally isomorphic to the identity on $\mathscr{L}$. This follows from the fact that if F^* is $\mathbb{A}^1$-local, then CF^* is isomorphic to F^* (see [MVW, 14.9]). $\square$

Example 4.1.9. An important class of examples is provided by the *geometric objects*. Let X be a smooth scheme. Then we may regard $\mathbb{Z}_{\mathrm{tr}}(X)$ as a cochain complex of Nisnevich sheaf with transfers concentrated in degree 0 (see Example 3.1.12); it represents an object in $\mathbf{D^-ST}$, and thus also an object in $\mathbf{DM}^{\mathrm{eff},-}$. We call the full triangulated subcategory of $\mathbf{DM}^{\mathrm{eff},-}$ generated by $\mathbb{Z}_{\mathrm{tr}}(X)$, as X ranges over all smooth schemes, the *effective geometric motives*, which we represent by $\mathbf{DM}^{\mathrm{eff},-}_{gm}$. We write $M(X)$ for the class of $\mathbb{Z}_{\mathrm{tr}}(X)$ in $\mathbf{DM}^{\mathrm{eff},-}$.

On the other hand, $C_*\mathbb{Z}_{\mathrm{tr}}(X)$ represents an object in $\mathscr{L}$, and we can similarly define the geometric objects of $\mathscr{L}$ as those belonging to the thick subcategory generated by the cochain complexes $C_*\mathbb{Z}_{\mathrm{tr}}(X)$, for X in Sm_k. By Theorem 4.1.8, $\mathbf{DM}_{gm}^{\mathrm{eff},-}$ corresponds to the geometric objects of $\mathscr{L}$.

4.2 Triangulated Monoidal Structure on $\mathbf{DM}^{\mathrm{eff},-}$

Recall from [Kelly82, 1.13] and [MVW, 8A.1] the notion of a symmetric closed monoidal structure generalized to the setting of a triangulated category:

Definition 4.2.1. Let $(\mathbf{D}, \otimes, \mathbb{1})$ be a triangulated category. We say that $\mathbf{D}$ is a *tensor triangulated category* if there exists a pair of natural isomorphisms

$$(M[1]) \otimes N \xrightarrow{l_{M,N}} (M \otimes N)[1] \xleftarrow[r_{M,N}]{} M \otimes (N[1])$$

such that $(\mathbf{D}, \otimes)$ satisfies the axioms of a symmetric monoidal category, and the following two conditions hold

1. For any distinguished triangle $M' \longrightarrow M \longrightarrow M'' \xrightarrow{\delta} M'[1]$, and any N in $\mathbf{D}$, the following triangles are distinguished

$$M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \xrightarrow{l(\delta \otimes D)} (M' \otimes N)[1]$$

$$N \otimes M' \longrightarrow N \otimes M \longrightarrow N \otimes M'' \xrightarrow{r(D \otimes \delta)} (N \otimes M')[1]$$

2. For any M and N in $\mathbf{D}$, the following anti-commutes, i.e., $rl = -lr$:

$$\begin{array}{ccc} M[1] \otimes N[1] & \xrightarrow{\quad r \quad} & (M[1] \otimes N)[1] \\ \downarrow l & -1 & \downarrow l \\ (M \otimes D[1])[1] & \xrightarrow{\quad r \quad} & (M \otimes D)[2]. \end{array}$$

We say that $(\mathbf{D}, \otimes)$ is an *additive symmetric monoidal category* if

$$\left(\bigoplus M_i\right) \otimes N = \bigoplus_i (M_i \otimes N)$$

for all N in $\mathbf{D}$ and all families $\{M_i\}$ of objects of $\mathbf{D}$ such that the direct sum $\oplus_i M_i$ exists in $\mathbf{D}$.

Recall that for a symmetric monoidal category $(\mathscr{C}, \otimes, \mathbb{1})$, an internal hom in $\mathscr{C}$ is a bifunctor

$$\underline{\mathrm{Hom}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathscr{C},$$

such that for all C in $\mathscr{C}$, the endofunctor $\underline{\mathrm{Hom}}(C, -)$ is right adjoint to the functor $- \otimes C$; in this case, we say that $\mathscr{C}$ is a *closed monoidal category.* Not every symmetric monoidal category admits an internal hom, although it is possible for $- \otimes C$ to admit right adjoints for some objects C of $\mathscr{C}$. We introduce the following definition to describe this notion:

Definition 4.2.2. For a symmetric monoidal category $(\mathscr{C}, \otimes, \mathbb{1})$, we say that $\mathscr{C}$ *has a partial internal hom* if there exists a full subcategory $\mathscr{C}^{\mathrm{rep}}$ of $\mathscr{C}$ containing $\mathbb{1}$, and a bifunctor $\underline{\mathrm{Hom}}(-,-) : (\mathscr{C}^{\mathrm{rep}})^{op} \times \mathscr{C} \longrightarrow \mathscr{C}$ such that for all F in $\mathscr{C}^{\mathrm{rep}}$, $F \otimes -$ is left adjoint to $\underline{\mathrm{Hom}}(F, -)$.

We call $\mathscr{C}^{\mathrm{rep}}$ the *semi-representable objects* of $\mathscr{C}$, and $\underline{\mathrm{Hom}}$ the partial internal hom in $\mathscr{C}$. We call the pair $(\underline{\mathrm{Hom}}, \mathscr{C}^{\mathrm{rep}})$ the *partial internal hom structure on* $\mathscr{C}$.

Following [MVW], we show that $\mathbf{DM}^{\mathrm{eff},-}$ is equipped with an additive symmetric monoidal structure and a partial internal hom structure. Let us first define the tensor and internal hom operators on $\mathbf{PST}$. By Yoneda Lemma, $\mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \mathbb{Z}_{\mathrm{tr}}(Y)) \cong Cor(X, Y)$ for all X and Y in Sm_k, i.e., we can identify a morphism between representable presheaves as a finite correspondence. The tensor structure will be determined by the following requirements.

1. $\mathbb{Z}_{\mathrm{tr}}(X) \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr}}(Y) \stackrel{\mathrm{def}}{=} \mathbb{Z}_{\mathrm{tr}}(X \times Y)$,
2. for each map φ in $\mathrm{Hom}_{\mathbf{PST}}(\mathbb{Z}_{\mathrm{tr}}(X), \mathbb{Z}_{\mathrm{tr}}(Y))$, let W be its associated finite correspondence in $Cor(X, Y)$. Then $\varphi \otimes \mathbb{Z}_{\mathrm{tr}}(Z) : \mathbb{Z}_{\mathrm{tr}}(X) \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr}}(Z) \longrightarrow \mathbb{Z}_{\mathrm{tr}}(Y) \otimes^{\mathrm{tr}} \mathbb{Z}_{\mathrm{tr}}(Z)$ corresponds to the finite correspondence $W \times Z$.

It is clear that we can extend the bifunctor $\otimes^{\mathrm{tr}}$ to arbitrary direct sums of representable presheaves.

Next, for arbitrary presheaves with transfers F, G, let $P^* \longrightarrow F$ and $Q^* \longrightarrow G$ be resolutions by direct sums of representable functors of F and G respectively. We write

$F \otimes^{\mathbb{L}} G$ for the total complex of the double complex $P^* \otimes^{\mathrm{tr}} Q^*$. By the Comparison Theorem [Wei94, 2.26], any two projective resolutions are chain homotopy equivalent. Therefore, it is easy to see that up to chain homotopy equivalence, this is independent of the choice of P^* and Q^*.

In particular, $H^0(F \otimes^{\mathbb{L}} G)$ is well-defined. Define the tensor operation on **PST** to be

$$F \otimes G \stackrel{\mathrm{def}}{=} H^0(F \otimes^{\mathbb{L}} G),$$

and define the internal hom presheaf by

$$\underline{\mathrm{Hom}}(F, G) : X \mapsto \mathrm{Hom}_{\mathbf{PST}}(F \otimes \mathbb{Z}_{\mathrm{tr}}(X), G).$$

These operations define a closed monoidal structure on **PST**. That is, for all F in **PST**, the functor $F \otimes^{\mathbb{L}} -$ is adjoint to $\underline{\mathrm{Hom}}(-, F)$ (see [MVW, 8.3]).

Remark 4.2.3. Notice that the $\otimes$ structure defined is *not* the usual tensor product on presheaves of abelian groups. In particular, $\mathbb{Z}_{\mathrm{tr}}(X)(Z) \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathrm{tr}}(Y)(Z) \neq \mathbb{Z}_{\mathrm{tr}}(X \times Y)(Z)$, where $\otimes_{\mathbb{Z}}$ denotes the usual tensor product of abelian groups.

We now extend $\otimes$ to $\mathbf{D}^-\mathbf{PST}$. To do so, let F^* represent a bounded above cochain complex of presheaves with transfers. By [Wei94, 10.5.6], F^* is quasi-isomorphic to a projective complex P^*. In fact, we may assume that P^* is a complex such that P^i is a direct sum of representable presheaves.

Define $F^* \otimes^{\mathbb{L}} G^*$ to be the direct sum total complex associated with $P^* \otimes Q^*$, where P^* and Q^* are projective resolutions of F^* and G^* respectively. Notice that $F^* \otimes^{\mathbb{L}} G^*$ is defined up to unique quasi-isomorphism. In particular, $\otimes^{\mathbb{L}}$ is defined up to quasi-isomorphism as a bifunctor on $\mathbf{D}^-\mathbf{PST}$. Indeed, let F^* and F'^* be two quasi-isomorphic bounded above complexes in **PST**. Then for any bounded above cochain G^* in **PST**, $F^* \otimes^{\mathbb{L}} G^* \cong F'^* \otimes^{\mathbb{L}} G^*$. (see [MVW, 8.7]) To show that $\mathbf{D}^-\mathbf{PST}$ is equipped with a tensor triangulated structure, we make the following observation.

Let $Cor^{\oplus}$ denote the closure under direct sum of representable presheaves in **PST**. This is an additive category equipped with an additive symmetric monoidal structure. By [MVW, 8A.4], we see that the homotopy category $\mathbf{K}^-(Cor^{\oplus})$ is a tensor triangulated categegory. By arguments similar to those in [Wei94, 10.4.8], $\mathbf{D^-PST}$ is equivalent as a category to $\mathbf{K}^-(Cor^{\oplus})$. It follows that $\mathbf{D^-PST}$ is also a tensor triangulated category.

Next, we show that the tensor structure is preserved under Nisnevich sheafification.

Definition 4.2.4. Let F, G be Nisnevich sheaves with transfers. Define $F \otimes^{\mathrm{tr}}_{\mathrm{Nis}} G$ to be the Nisnevich sheafification of the presheaf $F \otimes^{\mathrm{tr}} G$. That is,

$$F \otimes^{\mathrm{tr}}_{\mathrm{Nis}} G \overset{\mathrm{def}}{=} a_{\mathrm{Nis}}(F \otimes^{\mathrm{tr}} G)$$

where a_{Nis} is the Nisnevich sheafification. We can extend $\otimes^{\mathrm{tr}}_{\mathrm{Nis}}$ to cochain complexes of Nisnevich sheaves. Let F^* and G^* be bounded above cochain complexes of Nisnevich sheaves with transfers. Define $F^* \otimes^{\mathbb{L}}_{\mathrm{Nis}} G^*$ to be the Nisnevich sheafification of the complex $F^* \otimes^{\mathbb{L}} G^*$:

$$F^* \otimes^{\mathbb{L}}_{\mathrm{Nis}} G^* \overset{\mathrm{def}}{=} a_{\mathrm{Nis}}(F^* \otimes^{\mathbb{L}} G^*).$$

This is well-defined up to quasi-isomorphism.

Remark 4.2.5. Fix F and G sheaves with transfers, and let P^* and Q^* be resolutions by sums of representables of F^* and G^* respectively. Since a_{Nis} is exact, $a_{\mathrm{Nis}}(F \otimes^{\mathbb{L}} G) = a_{\mathrm{Nis}}(\mathrm{Tot}(P^* \otimes^{\mathrm{tr}} Q^*)) = \mathrm{Tot}(P^* \otimes^{\mathrm{tr}}_{\mathrm{Nis}} Q^*)$.

We claim that $(\mathbf{D^-ST}, \otimes^{\mathbb{L}}_{\mathrm{Nis}})$ is an additive symmetric monoidal triangulated category. The proof depends on the following lemma.

Lemma 4.2.6 ([MVW] Prop. 8A.7)**.** *Let $\mathbf{D}$ be a tensor triangulated category, and let $\mathscr{W}$ be a collection of maps in $\mathbf{D}$ that is closed under $- \otimes N$ for every N in $\mathbf{D}$, i.e., if $M \longrightarrow M'$ is in $\mathscr{W}$ then so is $M \otimes N \longrightarrow M' \otimes N$. Then the localization $\mathscr{W}^{\mathbf{-1}}$ is also a tensor triangulated category.*

To proceed, we note that if F^* is quasi-isomorphic to F'^*, then for every bounded above complex G^* of sheaves with transfers, $F^* \otimes^{\mathbb{L}}_{\mathrm{Nis}} G^*$ is quasi-isomorphic to $F'^* \otimes^{\mathbb{L}}_{\mathrm{Nis}} G^*$ (see [MVW, 8.16]). Therefore, $\otimes^{\mathbb{L}}_{\mathrm{Nis}}$ is a well-defined bifunctor on $\mathbf{D^-ST}$. Finally, observe that $\mathbf{D^-ST}$ is equivalent to the category obtained from $\mathbf{D^-PST}$ by formally inverting morphisms of the form $F^* \longrightarrow F'^*$ such that $a_{\mathrm{Nis}}(F^*) \longrightarrow a_{\mathrm{Nis}}(F'^*)$ is an quasi-isomorphism. We obtain a tensor triangulated structure on $\mathbf{D^-ST}$ by Lemma 4.2.6.

In fact, the same argument shows that $\mathbf{DM}^{\text{eff},-}$ is equipped with a tensor triangulated structure. Recall that $\mathbf{DM}^{\text{eff},-}$ is equivalent to the category obtained from $\mathbf{D}^{-}\mathbf{ST}$ by formally inverting the $\mathbb{A}^1$-weak equivalences. If $\varphi : F^* \longrightarrow F'^*$ is an $\mathbb{A}^1$-weak equivalence, then by [MVW, 9.5] for all G^* in $\mathbf{D}^{-}\mathbf{ST}$, $\varphi \otimes^{\mathbb{L}}_{\text{Nis}} G^* : F^* \otimes^{\mathbb{L}}_{\text{Nis}} G^* \longrightarrow F'^* \otimes^{\mathbb{L}}_{\text{Nis}} G^*$ is an $\mathbb{A}^1$-weak equivalence. Hence, $\otimes^{\mathbb{L}}_{\text{Nis}}$ induces a triangulated tensor product on $\mathbf{DM}^{\text{eff},-}$. We represent the tensor product on $\mathbf{DM}^{\text{eff},-}$ by $\otimes^{L}$.

There also exists a tensor operation on $\mathscr{L}$, which is different from the one defined on its parent category $\mathbf{D}^{-}\mathbf{ST}$. For F^*, G^* in $\mathscr{L}$, we define $F^* \otimes_{\mathscr{L}} G^*$ to be the direct sum total complex

$$\mathrm{Tot}^{\oplus} C(F^* \otimes^{\mathbb{L}}_{\text{Nis}} G^*).$$

The tensor product $\otimes_{\mathscr{L}}$ is a triangulated tensor product by [MVW, 14.11], and the categorical equivalence $\pi : \mathscr{L} \longrightarrow \mathbf{DM}^{\text{eff},-}$ in Theorem 4.1.8 is an equivalence of tensor triangulated categories.

Let us now define the partial internal hom structure on $\mathbf{DM}^{\text{eff},-}$. We will do this by defining a partial internal hom structure on $\mathscr{L}$. This, in turn, is obtained from the partial internal hom structure on $\mathbf{D}^{-}\mathbf{ST}$.

Definition 4.2.7. Fix a bounded above complex of Nisnevich sheaves B^* and an injective Cartan-Eilenberg resolution $B^* \longrightarrow I^*$, which exists by [MVW, 6.19]. For X in Sm_k, define $\underline{\mathrm{RHom}}(\mathbb{Z}_{\text{tr}}(X), B^*)$ to be the complex of sheaves given by

$$\underline{\mathrm{RHom}}(\mathbb{Z}_{\text{tr}}(X), B^*)(U) = \mathrm{Hom}^*_{\mathbf{D}^{-}\mathbf{ST}}(\mathbb{Z}_{\text{tr}}(X \times U), I^*).$$

Notice that the cochain complex $\mathrm{Hom}^*_{\mathbf{D}^{-}\mathbf{ST}}(\mathbb{Z}_{\text{tr}}(X \times U), I^*)$ is defined up to unique quasi-isomorphism. Furthermore, by [Voe00, 3.2.9], $\mathbf{H}^k \underline{\mathrm{RHom}}(\mathbb{Z}_{\text{tr}}(X), B^*) = 0$ for all $k > \dim X + l$, where l is the smallest index such that $\mathbf{H}^0 B^*$ has non-vanishing cohomology. Hence $\underline{\mathrm{RHom}}(\mathbb{Z}_{\text{tr}}(X), B^*)$ is an object of $\mathbf{D}^{-}\mathbf{ST}$. We can extend $\underline{\mathrm{RHom}}$ in the first factor to the thick subcategory $\mathbf{D}^{-}\mathbf{ST}^{\text{rep}}$ of $\mathbf{D}^{-}\mathbf{ST}$ generated by the sheaves $\mathbb{Z}_{\text{tr}}(X)$ regarded as cochain complexes concentrated in degree 0.

The lemma below follows from the construction of $\underline{\mathrm{RHom}}$.

Lemma 4.2.8. *For all X in Sm_k, and all F^*, G^* in $\mathbf{D^-ST}$, we have the following adjunction*

$$\mathrm{Hom}_{\mathbf{D^-ST}}(F^* \otimes^{\mathbb{L}}_{\mathrm{Nis}} \mathbb{Z}_{\mathrm{tr}}(X), G^*) \cong \mathrm{Hom}_{\mathbf{D^-ST}}(F^*, \underline{\mathrm{RHom}}(\mathbb{Z}_{\mathrm{tr}}(X), G^*)).$$

We now define the partial internal hom structure for $\mathscr{L}$. By [MVW, 14.12], if a bounded above cochain complex F^* in $\mathbf{D^-ST}$ is $\mathbb{A}^1$-local, then for all X in Sm_k, the cochain complex $\underline{\mathrm{RHom}}(\mathbb{Z}_{\mathrm{tr}}(X), F^*)$ is also $\mathbb{A}^1$-local.

Definition 4.2.9. Fix X in Sm_k, and let F^* be a bounded above $\mathbb{A}^1$-local complex. We define $\underline{\mathrm{RHom}}_{\mathscr{L}}(C_*\mathbb{Z}_{\mathrm{tr}}(X), F^*)$ to be the chain complex of sheaves given by

$$\underline{\mathrm{RHom}}_{\mathscr{L}}(C_*\mathbb{Z}_{\mathrm{tr}}(X), F^*)(U) \overset{\mathrm{def}}{=} \mathrm{Hom}^*_{\mathbf{D^-ST}}(C_*\mathbb{Z}_{\mathrm{tr}}(X \times U), F^*).$$

As in Definition 4.2.7, we may extend the definition of $\underline{\mathrm{RHom}}_{\mathscr{L}}$ in the first factor to all objects in the thick subcategory of $\mathscr{L}$ generated by the cochain complexes $C\mathbb{Z}_{\mathrm{tr}}(X)$.

Recall from Proposition 4.1.7 that the functor $F^* \mapsto CF^*$ is left adjoint to the inclusion of $\mathscr{L}$. Hence, by the definition of $\underline{\mathrm{RHom}}_{\mathscr{L}}$ above, we have the following equality of endofunctors of $\mathscr{L}$:

$$\underline{\mathrm{RHom}}_{\mathscr{L}}(C\mathbb{Z}_{\mathrm{tr}}(X), -) = \underline{\mathrm{RHom}}(\mathbb{Z}_{\mathrm{tr}}(X), -).$$

To see that $\underline{\mathrm{RHom}}_{\mathscr{L}}$ defines a partial internal hom in $\mathscr{L}$, we need to verify that for all X in Sm_k, $\underline{\mathrm{RHom}}_{\mathscr{L}}(C_*\mathbb{Z}_{\mathrm{tr}}(X), -)$ is right adjoint to $- \otimes_{\mathscr{L}} C_*\mathbb{Z}_{\mathrm{tr}}(X)$. Let F^* and G^* be bounded above $\mathbb{A}^1$-local complexes. We have the following chain of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{L}}(F^* \otimes_{\mathscr{L}} C_*\mathbb{Z}_{\mathrm{tr}}(X), G^*) &\overset{(1)}{=} \mathrm{Hom}_{\mathbf{D^-ST}}(C(F^* \otimes^{\mathbb{L}}_{\mathrm{Nis}} C_*\mathbb{Z}_{\mathrm{tr}}(X)), G^*) \\ &\overset{(2)}{\cong} \mathrm{Hom}_{\mathbf{D^-ST}}(F^* \otimes^{\mathbb{L}}_{\mathrm{Nis}} \mathbb{Z}_{\mathrm{tr}}(X), G^*) \\ &\overset{(3)}{\cong} \mathrm{Hom}_{\mathbf{D^-ST}}(F^*, \underline{\mathrm{RHom}}(\mathbb{Z}_{\mathrm{tr}}(X), G^*) \\ &\overset{(4)}{\cong} \mathrm{Hom}_{\mathscr{L}}(F^*, \underline{\mathrm{RHom}}_{\mathscr{L}}(C_*\mathbb{Z}_{\mathrm{tr}}(X), G^*)), \end{aligned}$$

where F^* and G^* are bounded above $\mathbb{A}^1$-local complexes, and X is an arbitrary smooth scheme. The equality in (1) follows from the definition of $\otimes_{\mathscr{L}}$; (2) and (4) follow from the adjunction introduced in Proposition 4.1.7; and (3) follows from the adjunction established in Lemma 4.2.8.

Via the categorical equivalence between $\mathscr{L}$ and $\mathbf{DM}^{\text{eff},-}$, there exists a partial internal hom structure on $\mathbf{DM}^{\text{eff},-}$. We write $\otimes^L$ and $\underline{\text{RHom}}_{\mathbf{DM}^{\text{eff},-}}$ for the tensor and partial internal hom operators respectively. Here, the semi-representable objects are the geometric motives $\mathbf{DM}^{\text{eff},-}_{gm}$ defined in Example 4.1.9. We have just established the proposition below:

Proposition 4.2.10 ([MVW] 14.12)**.** *Let $\otimes^L$ and $\underline{\text{RHom}}_{\mathbf{DM}^{\text{eff},-}}$ be given as above. Then for all M in $\mathbf{DM}^{\text{eff},-}_{gm}$, $-\otimes^L M$ is left adjoint to $\underline{\text{RHom}}_{\mathbf{DM}^{\text{eff},-}}(M,-)$.*

Remark 4.2.11. Notice that $\underline{\text{RHom}}$ and $\underline{\text{RHom}}_{\mathscr{L}}$ do not define a closed monoidal structure on their respective categories, as they are only defined on the geometric objects.

4.3 The motivic complex $\mathbb{Z}(n)$

We now introduce an important set of objects in $\mathbf{DM}^{\text{eff},-}$. Let $\mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$ denote the cokernel of

$$\mathbb{Z} = \mathbb{Z}_{\text{tr}}(\operatorname{Spec} k) \longrightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^1 - 0)$$

given by $k[x, x^{-1}] \longrightarrow k$, induced by $x \mapsto 1$. Since $k \longrightarrow k[x, x^{-1}]$ defines a splitting $\mathbb{Z}_{\text{tr}}(\mathbb{A}^1 - 0) \cong \mathbb{Z}_{\text{tr}}(\mathbb{G}_m) \oplus \mathbb{Z}$, $\mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$ is also a Nisnevich sheaf with transfers.

More generally, let X be a smooth scheme, and let x be a k-point of X represented by $\operatorname{Spec} k \longrightarrow X$. We define the *pointed presheaf* $\mathbb{Z}_{\text{tr}}(X, x)$ as the cokernel of $x : \mathbb{Z} \longrightarrow \mathbb{Z}_{\text{tr}}(X)$, which defines a splitting of the structure map $\mathbb{Z}_{\text{tr}}(X) \longrightarrow \mathbb{Z}$. By the same reason as above, $\mathbb{Z}_{\text{tr}}(X, x)$ is also a Nisnevich sheaf.

If $\{\mathbb{Z}_{\text{tr}}(X_i, x_i) : i = 1, \ldots, n\}$ is a collection of pointed schemes, we define their *wedge sum* $\bigwedge_i^n \mathbb{Z}_{\text{tr}}(X_i, x_i)$ to be

$$\operatorname{cok}\left(\bigoplus_i \mathbb{Z}_{\text{tr}}(X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n) \xrightarrow{id \times \cdots \times x_i \times \cdots \times id} \mathbb{Z}_{\text{tr}}(X_1 \times \cdots \times X_n)\right).$$

By induction, $\bigwedge_i \mathbb{Z}_{\text{tr}}(X_i, x_i)$ is a direct summand of $\mathbb{Z}_{\text{tr}}(X_1 \times \cdots \times X_n)$ (see [MVW, 2.13]), and defines a Nisnevich sheaf.

In particular, for each nonnegative integer n, we can define $\bigwedge_{i=0}^n \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1 - 0, 1)$, which we view as an object of $\mathbf{DM}^{\mathrm{eff},-}$. In fact, these are geometric motives, i.e., objects in the subcategory $\mathbf{DM}^{\mathrm{eff},-}_{gm}$. We write this object as $\mathbb{Z}(n)$, which we call the n-th *motivic complex*. It is easy to see that $\mathbb{Z}(n) \otimes^L \mathbb{Z}(m) \cong \mathbb{Z}(n+m)$.

Remark 4.3.1. The careful reader may notice that in [MVW], the motivic complex $\mathbb{Z}(n)$ is defined to be $C_* \bigwedge_i^n \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1 - 0, 1)$, and not $\bigwedge_i^n \mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^1 - 0, 1)$ (see [MVW, 3.1]). However, notice that in $\mathbf{DM}^{\mathrm{eff},-}$, the two definitions of $\mathbb{Z}(n)$ are identified. This is a straightforward consequence of the fact that, for a sheaf with transfers F the cohomological inclusion $F \longrightarrow C_*F$ is an $\mathbb{A}^1$-weak equivalence (see [MVW, 9.15]).

Remark 4.3.2. Nisnevich motivic cohomology with integer coefficients is defined as

$$H^{p,q}(X) = \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}_{\mathrm{tr}}(X), \mathbb{Z}(q)[p]).$$

Notice that $\mathbb{Z}(1) \cong \mathcal{O}^*[-1]$ ([MVW, 4.1]). Therefore $H^{1,1}(X) = \mathcal{O}^*(X)$, and $H^{2,1} = \mathrm{Pic}(X)$. Furthermore, $H^{n,n}(\mathrm{Spec}\, F) = K_n^M(F)$ ([MVW, 5.1]).

More generally, we have

$$H^{n,i}(X) \cong CH^i(X, 2i - n)$$

where $CH^i(X,k)$ denote the k-th higher Chow group of X ([MVW, 19.1]).

4.4 Cancellation Theorem

We conclude this chapter with an important result, taken from [Voe02, Corollary 4.10]. To simplify notation, for M in $\mathbf{DM}^{\mathrm{eff},-}$, we write $M(1)$ for $M \otimes^L \mathbb{Z}(1)$, and M_{-1} for $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(1), M)$, and write $M(n)$ and M_{-n} for the n-th iterations of applying $- \otimes^L \mathbb{Z}(1)$ and $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(1), -)$ respectively to M.

As $\mathbb{Z}(n) \otimes^L \mathbb{Z}(1) = \mathbb{Z}(n+1)$, the functor given by $M \mapsto M(n)$ is equal to the functor $- \otimes^L \mathbb{Z}(n)$. Since right adjoints of the same functor are naturally isomorphic, $M \mapsto M_{-n}$ is naturally isomorphic to $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n), -)$. Furthermore, $\mathbb{Z}(1)$ is an object of $\mathbf{DM}^{\mathrm{eff},-}_{gm}$. Thus, $- \otimes^L \mathbb{Z}(1)$ is left adjoint to $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(1), -)$ by Proposition 4.2.10. More generally, $\mathbb{Z}(n)$ is an object of $\mathbf{DM}^{\mathrm{eff},-}_{gm}$ for all positive integer n. It follows that $- \otimes^L \mathbb{Z}(n)$ is left adjoint to $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n), -)$.

Theorem 4.4.1 (Cancellation)**.** *For any M and N in $\mathbf{DM}^{\text{eff},-}$,*

$$\operatorname{Hom}_{\mathbf{DM}^{\text{eff},-}}(M(1),N(1)) \cong \operatorname{Hom}_{\mathbf{DM}^{\text{eff},-}}(M,N).$$

In other words, tensoring with $\mathbb{Z}(1)$ is fully and faithful.

This statement can likewise be interpreted for the category $\mathscr{L}$. For F^* and G^* bounded above $\mathbb{A}^1$-local complexes, by abuse of notation, write $F^*(1)$ for $F \otimes^{\mathbb{L}}_{\text{Nis}} C_*\mathbb{Z}(1)$. By Theorems 4.4.1 and 4.1.8,

$$\operatorname{Hom}_{\mathbf{D}^-\mathbf{ST}}(F^* \otimes^{\mathbb{L}}_{\text{Nis}} C_*\mathbb{Z}(1), G^* \otimes^{\mathbb{L}}_{\text{Nis}} C_*\mathbb{Z}(1)) \cong \operatorname{Hom}_{\mathbf{D}^-\mathbf{ST}}(F^*, G^*).$$

This is the version of the statement that we will use in subsequent chapters. One important corollary of Theorem 4.4.1 is the following:

Corollary 4.4.2. *For each M in $\mathbf{DM}^{\text{eff},-}$ and each nonnegative integer n, the counit map*

$$M(n)_{-n} \longrightarrow M.$$

is an isomorphism, natural in M.

Proof. By Theorem 4.1.8, it suffices to verify the statement for $\mathbb{A}^1$-local complexes.

Let F^* be the bounded above $\mathbb{A}^1$-local complex corresponding to M. Notice that by the Cancellation Theorem, reinterpreted for $\mathbb{A}^1$-local complexes,

$$\underline{\operatorname{RHom}}(C_*\mathbb{Z}(1), C_*\mathbb{Z}(1) \otimes^{\mathbb{L}}_{\text{Nis}} F^*)(U) = \operatorname{RHom}(\mathbb{Z}_{\text{tr}}(U), F^*) = F^*(U)$$

for all U in Sm_k. It follows that $\underline{\operatorname{RHom}}(C_*\mathbb{Z}(1), C_*\mathbb{Z}(1) \otimes^{\mathbb{L}}_{\text{Nis}} F^*) \longrightarrow F^*$ is an isomorphism. The corollary now follows by induction on n. □

Chapter 5

Slice Filtration on $\mathbf{DM}^{\text{eff},-}$ and DM

In this chapter, we construct a sequence of subcategories on $\mathbf{DM}^{\text{eff},-}$ using the tensor and the partial internal hom structure on $\mathbf{DM}^{\text{eff},-}$ defined in the previous chapter (see Section 4.2). In order to be more precise, we introduce the following notion.

Definition 5.0.1. Let $\mathscr{A}$ be a category. A descending *weak filtration* of $\mathscr{A}$ is a ($\mathbb{Z}$-indexed) sequence of subcategories

$$\mathscr{A} \supseteq \cdots \supseteq \mathscr{A}_i \supseteq \mathscr{A}_{i+1} \supseteq \cdots$$

together with coreflection functors $\varphi_i : \mathscr{A} \longrightarrow \mathscr{A}_i$ for each i such that φ_i restricts to the identity on $\mathscr{A}_i$. One can similarly define ascending weak filtrations using reflections $\mathscr{A} \longrightarrow \varphi_n \mathscr{A}$. We will represent a weak filtration as $(\mathscr{A}_*, \varphi_*)$, where $\mathscr{A}_i$ are the subcategories and φ_i are the reflection/coreflections.

We say that a weak filtration $(\mathscr{A}_*, \varphi_*)$ is *degenerate* if all subcategories $\mathscr{A}_n$ are equal. If $\mathscr{A}$ has a zero object, then we say that $(\mathscr{A}_*, \varphi_*)$ is *trivial* if each $\mathscr{A}_n$ consists of only the zero object.

Remark 5.0.2. An $\mathbb{N}$-indexed descending weak filtration is just a $\mathbb{Z}$-indexed descending weak filtration such that $\mathscr{A}_j = \mathscr{A}$ for all $j \leq 0$. Likewise, an $\mathbb{N}$-indexed ascending weak filtration is an ascending weak filtration for which $\mathscr{A}_j = \mathscr{A}_0$ for all negative j.

We show that there are two $\mathbb{N}$-indexed weak filtrations — one ascending, one descending — on $\mathbf{DM}^{\text{eff},-}$ defined below in (5.1.1) and (5.1.2). The construction is based on the work of Voevodsky, Huber, and Kahn [HK06]. We then recall the definition of the derived category of motives $\mathbf{DM}$ in Definition 5.3.1, and extend the two weak filtrations on $\mathbf{DM}^{\text{eff},-}$ to $\mathbb{Z}$-indexed weak filtrations on $\mathbf{DM}$, defined below in (5.3.3) and (5.3.8). Aside from Lemmas 5.2.1 and 5.2.5 and Propositions 5.2.9 and 5.2.11, the content from the first two sections is taken from [HK06, §1]. The extensions of the filtrations to **DM** are new.

5.1 Slice filtration on $\mathbf{DM}^{\text{eff},-}$

To simplify notation, following Chapter 4, we write $M(n)$ for $M \otimes^L \mathbb{Z}(n)$, and M_{-n} for $\underline{\text{RHom}}_{\mathbf{DM}^{\text{eff},-}}(\mathbb{Z}(n), M)$. Furthermore, let L^n denote the functor $- \otimes^L \mathbb{Z}(n)$, and R^n denote the functor $\underline{\text{RHom}}_{\mathbf{DM}^{\text{eff},-}}(\mathbb{Z}(n), -)$. By convention, define L^0 and R^0 to be the identity functor. As we have noted in the paragraph preceding Theorem 4.4.1, (L^n, R^n) form an adjoint pair of triangulated functors for each natural numbers n.

We first describe the descending weak filtration on $\mathbf{DM}^{\text{eff},-}$. Fix a natural number n, let $\mathbf{DM}^{\text{eff},-}_{\geq n}$ be the full subcategory of objects of the form $M(n)$ for some M in $\mathbf{DM}^{\text{eff},-}$, and let $\mathbf{DM}^{\text{eff},-}_{<n}$ be the full subcategory of objects M such that $M_{-n} = 0$.

Since $M(n+1) = M(1)(n)$ and $M_{-n} = 0$ implies $M_{-(n+1)} = 0$, we have the following towers of subcategories:

$$\mathbf{DM}^{\text{eff},-} = \mathbf{DM}^{\text{eff},-}_{\geq 0} \supseteq \mathbf{DM}^{\text{eff},-}_{\geq 1} \supseteq \mathbf{DM}^{\text{eff},-}_{\geq 2} \supseteq \cdots \supseteq 0 \tag{5.1.1}$$

$$0 = \mathbf{DM}^{\text{eff},-}_{<0} \subseteq \mathbf{DM}^{\text{eff},-}_{<1} \subseteq \mathbf{DM}^{\text{eff},-}_{<2} \subseteq \cdots \subseteq \mathbf{DM}^{\text{eff},-} \tag{5.1.2}$$

For each n, $L^n R^n : \mathbf{DM}^{\text{eff},-} \longrightarrow \mathbf{DM}^{\text{eff},-}_{\geq n}$ is right adjoint to the inclusion of $\mathbf{DM}^{\text{eff},-}_{\geq n}$ into $\mathbf{DM}^{\text{eff},-}$ (see [HK06, 1.1]). Moreover, by Corollary 4.4.2, $R^n L^n \cong \text{id}$. Since an object M in $\mathbf{DM}^{\text{eff},-}_{\geq n}$ is of the form $M'(n)$ for some M' in $\mathbf{DM}^{\text{eff},-}$, and

$$L^n R^n M = L^n R^n M'(n) \cong L^n M' = M,$$

the functor $L^n R^n$ is naturally isomorphic to the identity on $\mathbf{DM}^{\text{eff},-}_{\geq n}$.

Definition 5.1.3. Following [HK06], let $\nu^{\geq n}$ denote the triangulated functor $L^n R^n$. Thus, $\nu^{\geq n} M = M_{-n}(n)$.

Furthermore, for each M in $\mathbf{DM}^{\text{eff},-}$, there exists some M' in $\mathbf{DM}^{\text{eff},-}$ such that there is a distinguished triangle:

$$\nu^{\geq n} M \longrightarrow M \longrightarrow M' \longrightarrow \nu^{\geq n} M[1]. \tag{5.1.4}$$

By [HK06, 1.4(i, ii)], M' is uniquely defined up to unique isomorphism, and there is a cohomological functor $\nu^{<n}$ given by $M \mapsto M'$, which is the left adjoint to the inclusion of $\mathbf{DM}^{\mathrm{eff},-}_{<n}$ in $\mathbf{DM}^{\mathrm{eff},-}$. If M is in $\mathbf{DM}^{\mathrm{eff},-}_{<n}$, then $\nu^{\geq n}M = M_{-n}(n) = 0$. Since (5.1.4) is distinguished, $M \cong \nu^{<n}M$. To show that this isomorphism is natural in M, we will prove the stronger result that (5.1.4) is natural in M.

Fix a map $f : M \longrightarrow M'$ in $\mathbf{DM}^{\mathrm{eff},-}$. By the naturality of the counit $\epsilon : \nu^{\geq n} \longrightarrow \mathrm{id}$, we have the following commutative square:

$$\begin{array}{ccc} \nu^{\geq n}M & \xrightarrow{\epsilon_M} & M \\ \downarrow{\scriptstyle \nu^{\geq n}f} & & \downarrow{\scriptstyle f} \\ \nu^{\geq n}M' & \xrightarrow{\epsilon_{M'}} & M'. \end{array}$$

Completing the rows of the square into distinguished triangles, we have the following commutative diagram

$$\begin{array}{ccccccc} \nu^{\geq n}M & \xrightarrow{\epsilon_M} & M & \longrightarrow & \nu^{<n}M & \longrightarrow & \nu^{\geq n}M[1] \\ \downarrow{\scriptstyle \nu^{\geq n}f} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow \\ \nu^{\geq n}M' & \xrightarrow{\epsilon_{M'}} & M' & \longrightarrow & \nu^{<n}M' & \longrightarrow & \nu^{\geq n}M'[1] \end{array}$$

Since $\nu^{<n}$ is left adjoint to the inclusion of $\mathbf{DM}^{\mathrm{eff},-}_{<n}$ into $\mathbf{DM}^{\mathrm{eff},-}$,

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff},-}}(M, \nu^{<n}M') \cong \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff},-}}(\nu^{<n}M, \nu^{<n}M').$$

Hence, the induced map from $\nu^{<n}M \longrightarrow \nu^{<n}M'$ (the dotted arrow in the diagram above) is $\nu^{<n}f$. We summarize the main results of the above discussion in the following proposition:

Proposition 5.1.5. *The tower of subcategories in* (5.1.1) *defines a descending weak filtration of* $\mathbf{DM}^{\mathrm{eff},-}$*, where the coreflection functors*

$$\nu^{\geq n} : \mathbf{DM}^{\mathrm{eff},-} \longrightarrow \mathbf{DM}^{\mathrm{eff},-}_{\geq n}$$

are defined by $M \mapsto M_{-n}(n)$*.*

Furthermore, the tower in (5.1.2) *defines an ascending weak filtration of* $\mathbf{DM}^{\mathrm{eff},-}$*, with reflection functors*

$$\nu^{<n} : \mathbf{DM}^{\mathrm{eff},-} \longrightarrow \mathbf{DM}^{\mathrm{eff},-}_{<n},$$

defined by sending M *to* M' *in the triangle given in* (5.1.4)*.*

We call the pair of weak filtrations associated with the towers in (5.1.1) and (5.1.2) the *slice filtration on* $\mathbf{DM}^{\text{eff},-}$.

Notice that, by replacing M by $\nu^{\geq n}M$ in (5.1.4), we get distinguished triangles for all positive integers m and n, all of which are natural in M:

$$\nu^{\geq n}\nu^{\geq m}M \longrightarrow \nu^{\geq m}M \longrightarrow \nu^{<n}\nu^{\geq m}M \longrightarrow \nu^{\geq n}\nu^{\geq m}M[1]. \tag{5.1.6}$$

5.2 Fundamental invariants of the slice filtration

In this section, following [HK06, 1.4 (iv, v)], we define the slice and fundamental invariant functors associated to the slice filtration on $\mathbf{DM}^{\text{eff},-}$. Before we do so, we will show that the functors $\nu^{\geq n}$ and $\nu^{<n}$ satisfy a number of properties described in Proposition 5.2.11. Lemmas 5.2.1 and 5.2.5 and Propositions 5.2.9 and 5.2.11 are new.

We first digress to discuss two results from category theory. For the following, let L and R be a pair of adjoint endofunctors on $\mathscr{C}$, and suppose that the unit $\eta : \mathrm{id} \longrightarrow RL$ is a natural isomorphism. Write L^n and R^n for the n-th iteration of L and R respectively. Since L and R are adjoint functors, so are L^n and R^n. Write ϵ^n for the counit $L^nR^n \longrightarrow \mathrm{id}$ and η^n for the unit $\mathrm{id} \longrightarrow R^nL^n$. In this case, η^n is also a natural isomorphism for each positive integer n.

Lemma 5.2.1. *For each positive integer n, the natural isomorphism $(L^{n+1}R^n\eta)^{-1}$: $L^{n+1}R^{n+1}L \longrightarrow L(L^nR^n)$ fits into the following commutative diagram of natural transformations:*

$$\begin{array}{ccc} L^{n+1}R^{n+1}L & \xrightarrow{\epsilon^{n+1}L} & L \\ \downarrow & & \| \\ L(L^nR^n) & \xrightarrow{L\epsilon^n} & L. \end{array} \tag{5.2.2}$$

Dually, the natural isomorphism ηL^nR^{n+1} : $L^nR^nR \longrightarrow R(L^{n+1}R^{n+1})$ fits into the following commutative diagram of natural transformations:

$$\begin{array}{ccc} L^nR^nR & \xrightarrow{\epsilon^nR} & R \\ \downarrow{\scriptstyle \eta} & & \| \\ R(L^{n+1}R^{n+1}) & \xrightarrow{R\epsilon^{n+1}} & R. \end{array} \tag{5.2.3}$$

Proof. We first show that (5.2.2) is commutative. To do so, we proceed by induction on n. For the case $n = 0$, by the counit-unit adjunction, the following composition is the identity transformation:

$$L \xrightarrow{L\eta} LRL \xrightarrow{\epsilon L} L.$$

Therefore, $\epsilon L = L(\eta^{-1})$, and the following diagram commutes:

$$\begin{array}{ccc} LRL & \xrightarrow{\epsilon L} & L \\ \downarrow{\scriptstyle L(\eta^{-1})} & & \| \\ L & = & L. \end{array}$$

Now assume that for some integer n, the following diagram is commutative:

$$\begin{array}{ccc} L^n R^n L & \xrightarrow{\epsilon^n L} & L \\ \downarrow{\scriptstyle L^n R^{n-1} \eta^{-1}} & & \| \\ L(L^{n-1}R^{n-1}) & \xrightarrow{L\epsilon^{n-1}} & L. \end{array} \tag{5.2.4}$$

Write ϵ' for the natural transformation $L^n \epsilon R^n : L^n R^n \longrightarrow L^{n-1}R^{n-1}$. Applying the naturality of ϵ' to the natural isomorphism $\eta^{-1} : RL \longrightarrow \mathrm{id}$, we have the following commutative diagram

$$\begin{array}{ccc} L^n R^n RL & \xrightarrow{\epsilon' RL} & L^{n-1}R^{n-1}RL \\ \downarrow{\scriptstyle L^n R^n \eta^{-1}} & & \downarrow{\scriptstyle L^{n-1}R^{n-1}\eta^{-1}} \\ L^n R^n & \xrightarrow{\epsilon'} & L^{n-1}R^{n-1}. \end{array}$$

Now apply L to the above, we have

$$\begin{array}{ccc} L^{n+1}R^{n+1}L & \xrightarrow{L\epsilon' RL} & L^n R^n L \\ \downarrow{\scriptstyle L^{n+1}R^n \eta^{-1}} & & \downarrow{\scriptstyle L^n R^{n-1}\eta^{-1}} \\ L^{n+1}R^n & \xrightarrow{L\epsilon'} & L^n R^{n-1}, \end{array}$$

which fits together with (5.2.4) to give the following commutative diagram:

$$\begin{array}{ccccc} L^{n+1}R^{n+1}L & \xrightarrow{L\epsilon' RL} & L^n R^n L & \xrightarrow{\epsilon^n} & L \\ \downarrow{\scriptstyle L^{n+1}R^n(\eta^{-1})} & & \downarrow{\scriptstyle L^n R^{n-1}(\eta^{-1})} & & \| \\ L^{n+1}R^n & \xrightarrow{L\epsilon'} & L^n R^{n-1} & \xrightarrow{L\epsilon^{n-1}} & L. \end{array}$$

Notice that $\epsilon^n \circ L\epsilon' R = \eta^{n+1}$ and $\epsilon^{n-1} \circ \eta' = \epsilon^n$. Therefore, in the diagram above, the composition of the two maps in the top row is precisely $\epsilon^{n+1}L$ and the composition of in the bottom row is precisely $L\epsilon^n$. By induction, the commutativity of (5.2.2) is established. The commutativity of (5.2.3) follows by similar arguments. □

Lemma 5.2.5. *For all positive integers n and m, there exists a natural isomorphism $\tau : L^nR^nL^mR^m \longrightarrow L^mR^mL^nR^n$ such that the following is a commutative diagram of natural transformations:*

$$\begin{array}{ccc} L^nR^nL^mR^m & \xrightarrow{L^nR^n\epsilon^m} & L^nR^n \\ \downarrow{\scriptstyle \tau} & & \downarrow \\ L^mR^mL^nR^n & \xrightarrow{\epsilon^m L^nR^n} & L^nR^n \end{array} \tag{5.2.6}$$

Proof. We first consider the case $m \leq n$. By the counit-unit adjunction, the composition

$$R^m \xrightarrow{\eta^m R^m} R^mL^mR^m \xrightarrow{R^m\epsilon^m} R^m$$

is the identity transformation. Applying L^nR^{n-m} to the above, we obtain the following commutative square:

$$\begin{array}{ccc} L^nR^{n-m}R^mL^mR^m & \xrightarrow{L^nR^n\epsilon^m} & L^nR^{n-m}R^m \\ \downarrow{\scriptstyle (L^nR^{n-m}\eta^mR^m)^{-1}} & & \| \\ L^nR^{n-m}R^m & = & L^nR^n. \end{array}$$

Similarly, by the unit-counit adjunction, the compositions

$$L^m \xrightarrow{L^m\eta^m} L^mR^mL^m \xrightarrow{\epsilon^m L^m} L^m$$

is also the identity transformation. Applying the above to $L^{n-m}R^n$, we obtain the following commutative square:

$$\begin{array}{ccc} L^mL^{n-m}R^n & = & L^nR^n \\ \downarrow{\scriptstyle (L^m\eta^nL^{n-m}R^n)} & & \| \\ L^mR^mL^mL^{n-m}R^n & \xrightarrow{\epsilon^mL^nR^n} & L^mL^{n-m}R^n \end{array}$$

Combining these squares, and setting

$$\tau \stackrel{\text{def}}{=} (L^m\eta^nL^{n-m}R^n) \circ (L^nR^{n-m}\eta^mR^m)^{-1},$$

we obtain the commuting square (5.2.6) for $n \geq m$.

For the case $n < m$, iterating on the results of Lemma 5.2.1, we have the following commutative diagrams:

$$\begin{array}{ccc} R^nL^mR^m & \xrightarrow{R^n\epsilon^m} & R^n \\ \downarrow{\scriptstyle (\eta^nL^{m-n}R^{m-n})^{-1}} & & \| \\ L^{m-n}R^m & \xrightarrow{\epsilon^{m-n}R^n} & R^n \end{array} \tag{5.2.7}$$

and

$$\begin{array}{ccc} L^{m+n}R^{m+n}L^n & \xrightarrow{\epsilon^mL^n} & L^n \\ \downarrow{\scriptstyle L^mR^{m-n}\eta^n} & & \| \\ L^nL^{m-n}R^{m-n} & \xrightarrow{L^n\epsilon^{m-n}} & L^n. \end{array} \tag{5.2.8}$$

Applying (5.2.8) to R^n and L^n to (5.2.7), the resulting diagrams fit together to give

$$\begin{array}{ccc} L^nR^nL^mR^m & \xrightarrow{L^nR^n\epsilon^m} & L^nR^n \\ \downarrow{\scriptstyle (L^n\eta^nL^{m-n}R^{m-n})^{-1}} & & \| \\ L^nL^{m-n}R^{m-n}R^n & \longrightarrow & L^nR^n \\ \downarrow{\scriptstyle L^mR^{m-n}\eta^nR^n} & & \| \\ L^mR^mL^nR^n & \xrightarrow{\epsilon^mL^nR^n} & L^nR^n. \end{array}$$

By setting

$$\tau \overset{\text{def}}{=} (L^n\eta^nL^{m-n}R^{m-n})^{-1} \circ L^mR^{m-n}\eta^nR^n,$$

we see that (5.2.6) is commutative, and the lemma is established. □

Applying Lemma 5.2.5 to the pair of adjoint functors $M \mapsto M(n)$ and $M \mapsto M_{-n}$ on $\mathbf{DM}^{\text{eff},-}$, we obtain the following proposition:

Proposition 5.2.9. *There is a natural isomorphism $\nu^{\geq n}\nu^{\geq m} \xrightarrow{\tau} \nu^{\geq m}\nu^{\geq n}$ fitting into the following commutative diagram of natural transformations:*

$$\begin{array}{ccc} \nu^{\geq n}\nu^{\geq m} & \xrightarrow{\nu^{\geq n}\epsilon^m} & \nu^{\geq n} \\ \downarrow{\scriptstyle \tau} & & \| \\ \nu^{\geq m}\nu^{\geq n} & \xrightarrow{\epsilon^m\nu^{\geq n}} & \nu^{\geq n}, \end{array} \tag{5.2.10}$$

where $\epsilon^m : \nu^{\geq m} \longrightarrow \mathrm{id}$ is the unit. Furthermore, $\nu^{\geq n}\epsilon^m$ and $\epsilon^m\nu^{\geq n}$ are natural isomorphisms.

Proposition 5.2.11. *For all nonnegative integers m, n, such that $m \leq n$, and for all M in $\mathbf{DM}^{\text{eff},-}$, there exists the following natural isomorphisms:*

1. $\nu^{\geq m}\nu^{<n} \cong \nu^{<n}\nu^{\geq m}$.

2. $\nu^{\geq n}\nu^{<m} = \nu^{<m}\nu^{\geq n} = 0$.

3. $\nu^{<m}\nu^{<n} \cong \nu^{<n}\nu^{<m} \cong \nu^{<m}$.

4. $(\nu^{\geq n}M)(k) = \nu^{\geq n+k}M(k)$ *for all positive integers* k.

Proof. For part (1), apply the commutative diagram of functors (5.2.10) in Proposition 5.2.9 to an object M of $\mathbf{DM}^{\text{eff},-}$, and extend the rows to triangles. We obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\nu^{\geq n}\nu^{\geq m}M & \longrightarrow & \nu^{\geq m}M & \longrightarrow & \nu^{<n}\nu^{\geq m}M & \xrightarrow{+1} & \nu^{\geq n}\nu^{\geq m}M[1] \\
\downarrow\cong & & \| & & \vdots & & \downarrow\cong \\
\nu^{\geq m}\nu^{\geq n}M & \longrightarrow & \nu^{\geq m}M & \longrightarrow & \nu^{\geq m}\nu^{<n}M & \xrightarrow{+1} & \nu^{\geq m}\nu^{\geq n}M[1].
\end{array}$$

By the Five Lemma ([Wei94, 10.2.2]), we have that

$$\nu^{\geq m}\nu^{<n}M \cong \nu^{<n}\nu^{\geq m}M.$$

Since the rows are functorial in M and the isomorphism $\nu^{\geq n}\nu^{\geq m}M \longrightarrow \nu^{\geq m}\nu^{\geq n}M$ is natural, for a given map $f : M \longrightarrow M'$, the induced maps $\nu^{<n}\nu^{\geq m}(f)$ and $\nu^{\geq m}\nu^{<n}(f)$ fit into the following commutative square:

$$\begin{array}{ccc}
\nu^{<n}\nu^{\geq m}M & \xrightarrow{\nu^{<n}\nu^{\geq m}(f)} & \nu^{<n}\nu^{\geq m}M' \\
\downarrow\cong & & \downarrow\cong \\
\nu^{\geq m}\nu^{<n}M & \xrightarrow{\nu^{\geq m}\nu^{<n}(f)} & \nu^{\geq m}\nu^{<n}M'
\end{array}$$

Therefore, $\nu^{\geq m}\nu^{<n}$ is naturally isomorphic to $\nu^{<n}\nu^{\geq m}$.

Since $(\nu^{<m})_{-n} = 0$, it is clear that $\nu^{\geq n}\nu^{<m} = 0$. On the other hand, by Proposition 5.2.9, $\nu^{\geq m}\nu^{\geq n} = \nu^{\geq n}$. From the following functorial distinguished triangle

$$\nu^{\geq m}\nu^{\geq n} \longrightarrow \nu^{\geq n} \longrightarrow \nu^{<m}\nu^{\geq n} \longrightarrow \nu^{\geq m}\nu^{\geq n}[1]$$

it follows that $\nu^{<n}\nu^{\geq m} = 0$, which proves (2).

For (3), apply the slice triangle (5.1.4) to $\nu^{<n}M$ to obtain:

$$\nu^{\geq m}\nu^{<n}M \longrightarrow \nu^{<n}M \longrightarrow \nu^{<m}\nu^{<n}M \longrightarrow \nu^{\geq m}\nu^{<n}M[1].$$

Applying $\nu^{<n}$ to the slice triangle of M gives:

$$\nu^{<n}\nu^{\geq m}M \longrightarrow \nu^{<n}M \longrightarrow \nu^{<n}\nu^{<m}M \longrightarrow \nu^{<n}\nu^{\geq m}M[1],$$

and by part (2), there exists a natural isomorphism $\nu^{\geq m}\nu^{<n}M \longrightarrow \nu^{<n}\nu^{\geq m}$ which fits into the following commutative diagram:

$$\begin{array}{ccccccc}
\nu^{\geq m}\nu^{<n}M & \longrightarrow & \nu^{<n}M & \longrightarrow & \nu^{<m}\nu^{<n}M & \xrightarrow{+1} & \nu^{\geq m}\nu^{<n}M[1] \\
\downarrow \cong & & \| & & \vdots & & \downarrow \cong \\
\nu^{<n}\nu^{\geq m}M & \longrightarrow & \nu^{<n}M & \longrightarrow & \nu^{<n}\nu^{<m}M & \xrightarrow{+1} & \nu^{<n}\nu^{\geq m}M[1].
\end{array}$$

The fact that $\nu^{<n}\nu^{<m}M \cong \nu^{<m}\nu^{<n}M$ follows from the Five Lemma. Naturality in M now follows from part (1).

For part (4), the case $k = 1$ is established in [HK06, 1.4(v)]. The general case follows by induction on k. □

Setting $m = n-1$ in (5.1.6), we obtain the following functorial distinguished triangle:

$$\nu^{\geq n} \longrightarrow \nu^{\geq n-1} \longrightarrow \nu^{<n}\nu^{\geq n-1} \longrightarrow \nu^{\geq n}[1]. \tag{5.2.12}$$

Definition 5.2.13. For M in $\mathbf{DM}^{\mathrm{eff},-}$ and positive integer n, we say $\nu^{<n+1}\nu^{\geq n}M$ is the *n-th slice* of M, written as $\nu^n M$. Since $\nu^{<n}$ and $\nu^{\geq n}$ are triangulated functors, so is ν^n. We define the 0-th slice functor to be $\nu^{<0}$.

By Proposition 5.2.11(1), $\nu^n \cong \nu^{\geq n}\nu^{<n-1}$. In particular, the image of ν^n is in $\mathbf{DM}^{\mathrm{eff},-}_{\geq n}$. That is, for each M in $\mathbf{DM}^{\mathrm{eff},-}$, there exists some M' such that $\nu^n M \cong M'(n)$. Setting $M'' \stackrel{\mathrm{def}}{=} M'[-2n]$, we obtain the following proposition:

Proposition 5.2.14 ([HK06], 1.4(v))**.** *For each n and M, $\nu^n M = M''(n)[2n]$ for some unique M'' in $\mathbf{DM}^{\mathrm{eff},-}$. The object M'' is defined up to unique isomorphism.*

Definition 5.2.15. Following *loc. cit.*, we call M'' in Proposition 5.2.14 the *n-th fundamental invariant of M*, which we write as $c_n M$. For each positive n, c_n is an endofunctor on $\mathbf{DM}^{\mathrm{eff},-}$.

Example 5.2.16. It is clear that $\mathbb{Z}(n)$ is its own n-th slice. Furthermore, since $M(\mathbb{P}^n) = \oplus_{i=0}^{n}\mathbb{Z}(n)[2n]$ (see [MVW, 15.5]), it is easy to verify that

$$\nu^{<k}M(\mathbb{P}^n) = \begin{cases} M(\mathbb{P}^k) & \text{if } k \leq n \\ M(\mathbb{P}^n) & \text{otherwise} \end{cases}$$

and

$$\nu^{\geq k}M(\mathbb{P}^k) = \begin{cases} M(\mathbb{P}^{n-k})(k)[2k] & \text{if } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\nu^k M(\mathbb{P}^n) = \mathbb{Z}(k)[2k]$, and the k-th fundamental invariant of $M(\mathbb{P}^n)$ is $c_k M(\mathbb{P}^n) = \mathbb{Z}$, for $k = 0, 1, \ldots, n$.

5.3 Slice filtration on DM

In this section, we will extend the slice filtration on $\mathbf{DM}^{\text{eff},-}$ to $\mathbb{Z}$-indexed filtrations on $\mathbf{DM}$. Recall from [MVW, 14.2] the following definition of the category $\mathbf{DM}$.

Definition 5.3.1. Let $\mathbf{DM}$ be the category obtained from $\mathbf{DM}^{\text{eff},-}$ by inverting the operation $M \mapsto M(1)$. That is, the objects of $\mathbf{DM}$ are pairs (M, n), where M is an object of $\mathbf{DM}^{\text{eff},-}$, and n is any integer, such that $(M(1), n) \cong (M, n+1)$; the set of morphism between (M, n) and (M', n') is

$$\varinjlim_k \text{Hom}_{\mathbf{DM}^{\text{eff},-}}(M(k+n), M'(k+n')).$$

as k ranges over all integer values for which $k+n$ and $k+n'$ are positive. We write $\text{Hom}_{\mathbf{DM}}((M, n), (M', n'))$ for the hom set of (M, n) and (M', n').

By induction, we have that $(M, n) \cong (M \otimes^L \mathbb{Z}(n), 0)$, for any positive integer n and all M in $\mathbf{DM}^{\text{eff},-}$. In particular, if $(M, n) \cong (M', n')$ for $n \geq n'$, then $M \cong M'(n-n')$.

Furthermore, by the Cancellation Theorem (Theorem 4.4.1),

$$\text{Hom}_{\mathbf{DM}^{\text{eff},-}}(M, M') = \text{Hom}_{\mathbf{DM}^{\text{eff},-}}(M(n), M'(n))$$

for all positive integers. Therefore, the colimit in the definition of $\text{Hom}_{\mathbf{DM}}$ is a finite limit. That is, it suffices to take $k > |n| + |n'|$, say.

By the Cancellation Theorem 4.4.1, the localization functor $\Sigma^\infty : \mathbf{DM}^{\text{eff},-} \longrightarrow \mathbf{DM}$, given by sending M in $\mathbf{DM}^{\text{eff},-}$ to $(M, 0)$ is fully faithful. Therefore, we can identify $\mathbf{DM}^{\text{eff},-}$ as a full subcategory of $\mathbf{DM}$.

We will now give a description of the subcategories in the slice filtration on $\mathbf{DM}$. Each subcategory in the slice filtration will be full, and we describe only the objects in these subcategories.

Definition 5.3.2. For each integer k, let the objects of $\mathbf{DM}_{\geq k}$ consist of objects (M, n) for which $n \geq k$. As defined, $\mathbf{DM}_{\geq n+1} \subseteq \mathbf{DM}_{\geq n}$ and therefore, we have the following tower of subcategories:

$$\mathbf{DM} \supseteq \cdots \mathbf{DM}_{\geq -1} \supseteq \mathbf{DM}_{\geq 0} \supseteq \mathbf{DM}_{\geq 1} \supseteq \cdots . \tag{5.3.3}$$

Notice that for $n \geq 0$, $(M, n) \cong (M(n), 0)$. Therefore, if $M \cong M'(n)$ for some M' in $\mathbf{DM}^{\text{eff},-}$, $(M, 0) \cong (M', n)$ in $\mathbf{DM}$. Conversely, if $(M, 0)$ is in $\mathbf{DM}_{\geq n}$, then $(M, 0) \cong (M', n)$ for some M' in $\mathbf{DM}^{\text{eff},-}$. Hence, $M \cong M'(n)$ in $\mathbf{DM}^{\text{eff},-}$. It follows that the image of $\mathbf{DM}^{\text{eff},-}_{\geq n}$ under Σ^∞ coincides with $\mathbf{DM}_{\geq n}$, when $n \geq 0$. In Definition 5.3.4, we define a way to associate every object (M, n) in $\mathbf{DM}$ with an object $\nu^{\geq 0}(M, n)$ in $\mathbf{DM}_{\geq 0}$, and in Proposition 5.3.6, we show that $\nu^{\geq 0}$ is right adjoint to Σ^∞. Therefore, we can realize $\mathbf{DM}^{\text{eff},-}$ as a coreflective subcategory of $\mathbf{DM}$.

To show that this tower of subcategories constitutes a weak filtration of $\mathbf{DM}$, we must construct an extension of the functors $\nu^{\geq k}$ of Definition 5.1.3. By convention, for all M in $\mathbf{DM}^{\text{eff},-}$, define $\nu^{\geq n} M$ to be M for all $n \leq 0$.

Definition 5.3.4. For any integer k and a given object (M, n) in $\mathbf{DM}$, we set

$$\nu^{\geq k}(M, n) \overset{\text{def}}{=} (\nu^{\geq k-n} M, n).$$

This definition preserves isomorphisms. Indeed, if $(M, n) \cong (M', n')$ for some integer n' less than n, say, then $M(n - n') = M'$, and $\nu^{\geq k-n'} M' \cong \nu^{\geq k-n} M(n - n')$ by (4) of Proposition 5.2.11. Hence, $(\nu^{\geq k-n'} M', n') \cong (\nu^{\geq k-n} M, n)$.

We want to show that the $\nu^{\geq k}$ are triangulated functors from $\mathbf{DM}$ to $\mathbf{DM}_{\geq k}$ that make $(\mathbf{DM}_{\geq k}, \nu^{\geq k})$ into a weak filtration. We will verify this claim in the Proposition 5.3.6. Let us first prove the following lemma:

Lemma 5.3.5. *If $(M_1, n) \longrightarrow (M_2, n) \longrightarrow (M_3, n_3) \longrightarrow (M_1, n)[1]$ is a distinguished triangle in* **DM***, then there exists some M such that $(M, n) \cong (M_3, n_3)$.*

Proof. Let φ denote the map from (M_1, n) to (M_2, n). Then φ is identified with some map $\varphi' : M_1 \longrightarrow M_2$ in $\mathbf{DM}^{\text{eff},-}$. Complete φ' to a triangle:

$$M_1 \longrightarrow M_2 \longrightarrow M \longrightarrow M_1[1].$$

Then, we have

$$\begin{array}{ccccccc} (M_1, n) & \xrightarrow{\varphi} & (M_2, n) & \longrightarrow & (M_3, n_3) & \longrightarrow & (M_1, n)[1] \\ \| & & \| & & \downarrow & & \| \\ (M_1, n) & \xrightarrow{\varphi} & (M_2, n) & \longrightarrow & (M, n) & \longrightarrow & (M_1, n)[+1]. \end{array}$$

The claim now follows from the Five Lemma. ☐

Proposition 5.3.6. *Let k be an arbitrary integer.*

1. *$(M, n) \mapsto \nu^{\geq k}(M, n)$ defines a triangulated functor.*

2. *$\nu^{\geq k}$ is a right adjoint to the inclusion of $\mathbf{DM}_{\geq k}$ into* **DM***.*

3. *the restriction of $\nu^{\geq k}$ to $\mathbf{DM}_{\geq k}$ is naturally isomorphic to the identity.*

Proof. If $k \leq n$, then $\nu^{\geq k}(M, n) = (M, n)$, and by definition (M, n) is an object of $\mathbf{DM}_{\geq k}$. On the other hand, if $k > n$, then as defined, $\nu^{\geq k}(M, n) = (\nu^{\geq k-n} M, n)$. By [HK06, 1.1], $\nu^{\geq k-n} M$ is in $\mathbf{DM}^{\text{eff},-}_{\geq k-n}$. Hence, $M \cong M'(k-n)$. Therefore, $\nu^{\geq k}(M, n) \cong (M'(k-n), n) \cong (M', k)$. This shows that $\nu^{\geq k}(M, n)$ is always an object of $\mathbf{DM}_{\geq k}$.

Consider a map $f : (M, n) \longrightarrow (M', n')$. Since we have already shown that $\nu^{\geq k}$ preserves isomorphisms, by replacing either (M, n) or (M', n') by an isomorphic object, we may assume that $n = n'$, and f comes from a map $g : M \longrightarrow M'$ in $\mathbf{DM}^{\text{eff},-}$. Define $\nu^{\geq k}(f)$ to be the map given by $\nu^{\geq k-n} g$ in $\mathbf{DM}^{\text{eff},-}$. This definition preserves the identity map, isomorphisms, and composition. It follows that $\nu^{\geq k}$ is a functor on **DM** whose image lies in $\mathbf{DM}_{\geq k}$.

Given a triangle,

$$(M', n') \longrightarrow (M, n) \longrightarrow (M'', n'') \longrightarrow (M', n')[1]$$

we may assume without loss of generality that $n = n' = n''$, and that this distinguished triangle comes from the distinguished triangle in $\mathbf{DM}^{\text{eff},-}$:

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow M'[1]$$

Since $\nu^{\geq k-n}$ is a triangulated functor on $\mathbf{DM}^{\text{eff},-}$ (see Definition 5.1.3), it follows that

$$\nu^{\geq k-n}M' \longrightarrow \nu^{\geq k-n}M \longrightarrow \nu^{\geq k-n}M'' \longrightarrow \nu^{\geq k-n}M'[1]$$

is a distinguished triangle in $\mathbf{DM}^{\text{eff},-}$. Thus, we have the following distinguished triangle in $\mathbf{DM}$:

$$\nu^{\geq k}(M', n) \longrightarrow \nu^{\geq k}(M, n) \longrightarrow \nu^{\geq k}(M'', n) \longrightarrow \nu^{\geq k}(M', n)[1].$$

Therefore, $\nu^{\geq k}$ is a triangulated functor, which proves part (1) of the proposition.

For part (2), let (M, n) be an object of $\mathbf{DM}$, and (M', n') be an object of $\mathbf{DM}_{\geq k}$. By replacing (M', n') with an isomorphic object, we may assume that $n' = k$. In the case $n > k$, notice that $\nu^{\geq k}(M, n) = (M, n)$, and the adjunction relation is trivially satisfied. Otherwise, for some suitably large integer l, we have the following equality:

$$\text{Hom}_{\mathbf{DM}}((M', k), (M, n)) = \text{Hom}_{\mathbf{DM}^{\text{eff},-}}(M'(l + k), M(l + n)).$$

Since $M'(l + k) \in \mathbf{DM}^{\text{eff},-}_{\geq k+l}$ and $\nu^{\geq k+l}$ is right adjoint to the inclusion of $\mathbf{DM}^{\text{eff},-}_{\geq l+k}$ into $\mathbf{DM}^{\text{eff},-}$,

$$\text{Hom}_{\mathbf{DM}^{\text{eff},-}}(M'(l + k), M(l + n)) \cong \text{Hom}_{\mathbf{DM}^{\text{eff},-}}(M'(l + k), \nu^{\geq k+l}M(l + n)).$$

Notice that by Proposition 5.2.11(4), $\nu^{\geq k+l}M(n + l) \cong (\nu^{\geq k-n}M)(n + l)$. Therefore,

$$\begin{aligned} \text{Hom}_{\mathbf{DM}}((M', k), (M, n)) &\cong \text{Hom}_{\mathbf{DM}^{\text{eff},-}}(M'(l + k), (\nu^{\geq k-n}M)(l + n)) \\ &\cong \text{Hom}_{\mathbf{DM}^{\text{eff},-}}((M', k), \nu^{\geq k}(M, n)). \end{aligned}$$

Since the isomorphism is functorial in both (M, n) and (M', k), it follows that $\nu^{\geq k}$ is right adjoint to the inclusion of $\mathbf{DM}_{\geq k}$ into $\mathbf{DM}$.

For part (3), if (M, n) is an object of $\mathbf{DM}_{\geq k}$, then $(M, n) \cong (M', k)$ for some M'. Furthermore, as defined, $\nu^{\geq k}(M', k) = (M', k)$. As this isomorphism is natural in (M, n), we have just established part (3). ∎

Next, we construct the ascending weak filtration on $\mathbf{DM}$.

Definition 5.3.7. Let $\mathbf{DM}_{<k}$ to be the full subcategory of objects (M,n) in $\mathbf{DM}$ for which $\nu^{\geq k}(M,n)=0$. Since $\nu^{\geq k}(M,n)=(\nu^{\geq k-n}M,n)=0$ implies that $\nu^{\geq k+1}(M,n)=(\nu^{\geq k+1-n}M,n)=0$, $\mathbf{DM}_{<k+1}$ is a subcategory of $\mathbf{DM}_{<k}$, and we obtain the following tower of subcategories:

$$0\subseteq\cdots\subseteq\mathbf{DM}_{<0}\subseteq\mathbf{DM}_{<1}\subseteq\mathbf{DM}_{<2}\subseteq\cdots\subseteq\mathbf{DM}. \tag{5.3.8}$$

As expected, the tower also defines a filtration of $\mathbf{DM}$, and to show this, we will define the reflection functors $\nu^{<k}:\mathbf{DM}\longrightarrow\mathbf{DM}_{<k}$. These reflection functors will come from extending the endofunctor $\nu^{<k}$ of $\mathbf{DM}^{\mathrm{eff},-}$ to endofunctors on $\mathbf{DM}$. By convention, for a nonpositive integer k, let us define the endofunctor $\nu^{<k}$ on $\mathbf{DM}^{\mathrm{eff},-}$ to be 0. For (M,n) in $\mathbf{DM}$, we have the following triangle

$$\nu^{\geq k}(M,n)\longrightarrow(M,n)\longrightarrow(M',n')\longrightarrow\nu^{\geq k}(M,n)[1]. \tag{5.3.9}$$

Lemma 5.3.10. *For each integer k, the object (M',n') in* (5.3.9) *is defined up to unique isomorphism. In particular, (M',n') is uniquely isomorphic to $(\nu^{<k-n}M,n)$.*

Proof. By definition, $\nu^{\geq k}(M,n)=(\nu^{\geq k-n}M,n)$. By Lemma 5.3.5 we may assume $n'=n$, and $\nu^{\geq k-n}M\longrightarrow M\longrightarrow M'\longrightarrow\nu^{\geq k-n}M[1]$ is a distinguished triangle in $\mathbf{DM}^{\mathrm{eff},-}$. Since M' is uniquely defined up to unique isomorphism (see [HK06, 1.3(i)]), $M'\cong\nu^{<k-n}M$, and $(M',n')\cong(\nu^{<k-n}M,n)$ as claimed. □

Recall the convention that for $k<0$, $\nu^{<k}=0$ as an endofunctor on $\mathbf{DM}^{\mathrm{eff},-}$. We now define the extension of $\nu^{<k}$ as endofunctors on $\mathbf{DM}$. As we will see in Proposition 5.3.12, the functors $\nu^{<k}$ are the reflection from $\mathbf{DM}$ to $k_{<*}\mathbf{DM}$.

Definition 5.3.11. For any integer k and a given object (M,n) in $\mathbf{DM}$, we set

$$\nu^{<k}(M,n)\overset{\mathrm{def}}{=}(\nu^{<k-n}M,n).$$

Copying the proof of [HK06, 1.3], $(M,n)\mapsto\nu^{<k}(M,n)$ defines a triangulated functor on $\mathbf{DM}$.

The arguments of [HK06, 1.3] can be adapted for $\nu^{<k}$ to show the following proposition.

Proposition 5.3.12. *For each integer k,*

1. *$\nu^{<k}$ is a triangulated functor*
2. *the image of $\nu^{<k}$ is $\mathbf{DM}_{<k}$ and $\nu^{<k}$ defines a left adjoint to the inclusion of $\mathbf{DM}_{<k}$ into $\mathbf{DM}$.*
3. *the restriction of $\nu^{<k}$ to $\mathbf{DM}_{<k}$ is naturally isomorphic to the identity.*
4. *If $k > 0$, the restriction of $\nu^{<k}$ to $\mathbf{DM}^{\text{eff},-}$ is the functor $\nu^{<k}$ of Proposition 5.1.5.*

It follows that the towers of subcategories given in (5.3.3) and (5.3.8) respectively define a descending and an ascending filtration on $\mathbf{DM}$.

5.4 Extending the fundamental invariants

We can also extend the definition of the fundamental invariants c_k to negative integers k. Notice that for each (M, n) in $\mathbf{DM}$, and each integer k, we have the slice triangle:

$$\nu^{\geq k+1}(M,n) \longrightarrow \nu^{\geq k}(M,n) \longrightarrow \nu^{<k+1}\nu^{\geq k}(M,n) \longrightarrow \nu^{\geq k+1}(M,n)[1].$$

Definition 5.4.1. Let us define ν^k to be $\nu^{<k+1}\nu^{\geq k}$. We call this functor the *k-th slice* on $\mathbf{DM}$. Since both $\nu^{<k+1}$ and $\nu^{\geq k}$ are triangulated functors, so is ν^k.

By arguments similar to those in the proof of Proposition 5.2.14, we have that $\nu^k(M,n) \cong (M''[2k], k)$ for some M'' in $\mathbf{DM}^{\text{eff},-}$, which is unique up to unique isomorphism. We define the *k-th fundamental invariant* of (M, n) to be

$$c_k(M,n) \stackrel{\text{def}}{=} M''.$$

For each integer k, c_k is a functor from $\mathbf{DM}$ to $\mathbf{DM}^{\text{eff},-}$.

Notice that for $k \geq 0$, if (M, n) is in $\mathbf{DM}^{\text{eff},-}$, the definition of ν^k recovers the k-th slice functor in Definition 5.2.13 by Proposition 5.3.12. Similarly, c_k is an extension of the k-th fundamental invariant on $\mathbf{DM}^{\text{eff},-}$.

We conclude this section by discussing the relationship between the tensor structure on $\mathbf{DM}^{\text{eff},-}$ and $\mathbf{DM}$ and their respective slice filtrations. First, let us define a tensor product on $\mathbf{DM}$. By abuse of notation, we will also represent this tensor product by $\otimes^L$.

Definition 5.4.2. Following [MVW, 8A], for objects $(M,n),(M',n')$ in $\mathbf{DM}$, we define $(M,n)\otimes^L(M',n')$ to be $(M\otimes^L M', n+n')$. As shown in [MVW, 15.8], the cyclic permutation of $\mathbb{Z}(1)^{\otimes 3}$ is the identity in $\mathbf{DM}^{\text{eff},-}$. By [MVW, 8A.12] the triangulated category $\mathbf{DM}$ together with $\otimes^L$ defines $(\mathbf{DM},\otimes^L)$ is an additive symmetric monoidal triangulated category.

Let us first consider the following result, which relate the slice filtration to the tensor product on $\mathbf{DM}^{\text{eff},-}$.

Proposition 5.4.3 ([HK06] 1.6)**.** *For nonnegative integers n,n', there exists a unique natural isomorphism $\eta:\nu^{\geq n}\otimes^L\nu^{\geq n'}\longrightarrow\nu^{\geq n+n'}(-\otimes^L-)$ compatible with the tensor structure on $\mathbf{DM}^{\text{eff},-}$. That is, we have the following commutative square for each M and M' in $\mathbf{DM}^{\text{eff},-}$:*

$$\begin{array}{ccc}
\nu^{\geq n}(M)\otimes^L\nu^{\geq n'}(M') & \xrightarrow{\eta} & \nu^{\geq n+n'}(M\otimes^L M') \\
\downarrow & & \downarrow \\
M\otimes^L M' & = & M\otimes^L M'.
\end{array}$$

We can extend this result to $\mathbf{DM}$. The following is a straightforward consequence of Proposition 5.4.3.

Corollary 5.4.4. *For all integers n,n', there exists a unique natural transformation of bifunctors on $\nu^{\geq n}\otimes^L\nu^{\geq n'}\longrightarrow\nu^{\geq n+n'}(-\otimes^L-)$ compatible with the tensor structure of* $\mathbf{DM}$.

A corollary of Proposition 5.4.3 applies to the tensor structure on the slices (and similarly, on the fundamental invariants) of the slice filtration.

Corollary 5.4.5. *For all integers nonnegative n,n', there exists unique natural transformations of bifunctors $\nu^n\otimes^L\nu^{n'}\longrightarrow\nu^{n+n'}(-\otimes^L-)$ and $c_n\otimes^L c_{n'}\longrightarrow c_{n+n'}(-\otimes^L-)$ compatible with the tensor structure on $\mathbf{DM}^{\text{eff},-}$.*

The natural transformations can be extended to natural transformations on the slice structure on $\mathbf{DM}$*: we have natural transformations* $\nu^n \otimes^L \nu^{n'} \longrightarrow \nu^{n+n'}(- \otimes^L -)$ *and* $c_n \otimes^L c_{n'} \longrightarrow c_{n+n'}(- \otimes^L -)$ *compatible with the tensor structure on* $\mathbf{DM}$.

Proof. The existence of natural transformations $\nu^n \otimes^L \nu^{n'} \longrightarrow \nu^{n+n'}(- \otimes^L -)$ and $c_n \otimes^L c_{n'} \longrightarrow c_{n+n'}(- \otimes^L -)$ on $\mathbf{DM}^{\mathrm{eff},-}$ is proven in [HK06, 1.6].

To show that the natural transformations are also defined on $\mathbf{DM}$, fix integers n, n', and let (M, m) and (M', m') be two objects in $\mathbf{DM}$. Since $(M', m') \cong (M(k)', m' - k)$, we may assume without loss of generality that $m = m' < \min(n, n', n + n')$. In this case, notice that

$$\nu^n(M, m) = (\nu^{n-m}M, m) \text{ and } \nu^{n'}(M', m) = (\nu^{n'-m}M', m),$$

and

$$c_n(M, m) = \nu^{n-m}(M)[-n] \text{ and } c_{n'}(M, m) = \nu^{n'-m}(M)[-n']$$

Define

$$\nu^n(M, m) \otimes^L \nu^{n'}(M', m) \longrightarrow \nu^{n+n'}((M, m) \otimes^L (M', m))$$

to be

$$(\nu^{n-m}(M) \otimes^L \nu^{n'-m}, 2m) \longrightarrow (\nu^{n+n'-2m}(M \otimes^L M'), 2m)$$

and

$$c_n(M, m) \otimes^L c_{n'}(M', m) \longrightarrow c_{n+n'}((M, m) \otimes^L (M', m))$$

to be

$$c_{n-m}M[-n] \otimes^L c_{n'-m}M'[-n'] \longrightarrow c_{n+n'-2m}M[-(n + n')].$$

Both maps are independent of the choice of m. Naturality in (M, m) and (M', m) follows from the naturality in M and M'. □

Remark 5.4.6. Notice that the fundamental invariants c_k of the slice filtration on $\mathbf{DM}$ always take value in $\mathbf{DM}^{\mathrm{eff},-}$. More specifically, the fundamental invariants always take value in the full subcategory of birational motives defined in [KaSu]. This is established for the fundamental invariants for $\mathbf{DM}^{\mathrm{eff},-}$ in [HK06, Section 2], and can be extended directly to $\mathbf{DM}$.

Chapter 6

Filtrations on HI

The purpose of this chapter is to construct three filtrations of **HI**. The main result of this chapter is that there is a sequence of coradicals (see Definition 2.1.3) on the category **HI** which induces a descending strong filtration and an ascending cofiltration (see Definition 6.2.5 below) of **HI** by the associated subcategories (see Theorem 6.2.10). The key ingredient in the constructions of the filtrations is the tensor monoidal structure on **HI** and the partial internal hom. These structures are induced by the tensor and partial internal hom operators on $\mathbf{DM}^{\mathrm{eff},-}$ introduced in Section 4.2. All uncredited results in this section are new.

6.1 Tensor and partial internal hom structure on HI

To simplify the definition and the proofs in this chapter and the next, we invoke Theorem 4.1.8 and identify the category $\mathbf{DM}^{\mathrm{eff},-}$ with the full triangulated subcategory $\mathscr{L}$ of $\mathbb{A}^1$-local complexes from Definition 4.1.4. We identify objects M in $\mathbf{DM}^{\mathrm{eff},-}$ with bounded above complexes F^* of Nisnevich sheaves with transfers such that H^nF^* is a homotopy invariant presheaf with transfers for every n. In particular, regarding a sheaf with transfers as a cochain complex concentrated in degree 0, we consider **HI** as an additive subcategory of $\mathbf{DM}^{\mathrm{eff},-}$.

Recall the following notions from [BBD, 1.3]:

Definition 6.1.1. A *t-category* is a triangulated category $\mathbf{D}$ together with a pair of full subcategories $(\mathbf{D}^{\geq 0}, \mathbf{D}^{\leq 0})$, called the *positive objects* and *negative object* of $\mathbf{D}$ respective, which satisfies the following properties:

1. For all X in $\mathbf{D}^{\leq 0}$, and Y in $\mathbf{D}^{\geq 1}$, $\mathrm{Hom}_{\mathbf{D}}(X, Y) = 0$.

2. $\mathbf{D}^{\leq 0} \subset \mathbf{D}^{\leq 1}$ and $\mathbf{D}^{\geq 1} \subset \mathbf{D}^{\geq 0}$

3. For all X in $\mathbf{D}$, there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

   such that A is in $\mathbf{D}^{\leq 0}$ and B is in $\mathbf{D}^{\geq 1}$.

Here we write $\mathbf{D}^{\geq n}$ and $\mathbf{D}^{\leq n}$ for $\mathbf{D}^{\geq 0}[n]$ and $\mathbf{D}^{\leq 0}[n]$ respectively. We call the pair $(\mathbf{D}^{\geq 0}, \mathbf{D}^{\leq 0})$ a *t-structure* on $\mathbf{D}$.

The *heart* of a t-category is the full subcategory $\mathscr{C} \stackrel{\text{def}}{=} \mathbf{D}^{\geq 0} \cap \mathbf{D}^{\leq 0}$.

If $\mathbf{D}$ is a t-category, then the inclusion of $\mathbf{D}^{\leq n}$ in $\mathbf{D}$ admits a right adjoint $\tau_{\leq n} : \mathbf{D} \longrightarrow \mathbf{D}^{\leq n}$, and the inclusion of $\mathbf{D}^{\geq n}$ in $\mathbf{D}$ admits left adjoint $\tau_{\geq n} : \mathbf{D} \longrightarrow \mathbf{D}^{\geq n}$. Furthermore, for all X in $\mathbf{D}$, there exists a unique map d in $\mathrm{Hom}_{\mathbf{D}}(\tau_{\geq 1}X, \tau_{\leq 0}X[1])$ such that

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \stackrel{d}{\longrightarrow} \tau_{\leq 0}X[1]$$

is distinguished (see [BBD, 1.3.3]). For integers m and n such that $m < n$, $\tau_{\leq m}\tau_{\leq n} = \tau_{\leq n}\tau_{\leq m} = \tau_{\leq m}$, and $\tau_{\geq m}\tau_{\geq n} = \tau_{\geq n}\tau_{\geq m} = \tau_{\geq n}$. Furthermore, $\tau_{\leq m}\tau_{\geq n} = \tau_{\geq n}\tau_{\leq m} = 0$, and $\tau_{\leq n}\tau_{\geq m} = \tau_{\geq m}\tau_{\leq n}$ (see [BBD, 1.3.5]). When $n = m = 0$, the composition $\tau_{\leq 0}\tau_{\geq 0}$ defines an additive functor $\mathbf{H}^0 : \mathbf{D} \longrightarrow \mathscr{C}$.

Recall from [BBD, 1.2.5] that an abelian subcategory $\mathscr{C}$ of $\mathbf{D}$ is *admissible* if for all C and D in $\mathscr{C}$ and $i < 0$, $\mathrm{Hom}_{\mathbf{D}}(C, D[i]) = 0$, and all exact sequences in $\mathscr{C}$ come from distinguished triangles in $\mathbf{D}$.

Theorem 6.1.2 ([BBD] 1.3.6)**.** *Let $\mathbf{D}$ be a t-category, and let $(\mathbf{D}^{\geq 0}, \mathbf{D}^{\leq 0})$ be its associated t structure. Then the heart $\mathscr{C}$ is an admissible abelian category, stable under taking extensions.*

Example 6.1.3 ([BBD] 1.3.2)**.** Let $\mathscr{A}$ be an abelian category, and $\mathbf{D}\mathscr{A}$ be its derived category. There is a natural t-structure on $\mathbf{D}\mathscr{A}$. The pair $(\mathbf{D}\mathscr{A}^{\geq 0}, \mathbf{D}\mathscr{A}^{\leq 0})$ is a pair of full subcategories whose objects are those with trivial cohomology in the negative and positive degrees respectively. In this case, the functors $\tau_{\geq n}$ and $\tau_{\leq n}$ are given by good truncations.

The heart of this t-structure is precisely $\mathscr{A}$, where an object of $\mathscr{A}$ is regarded as a complex concentrated in degree 0. (See the example following the statement of 1.3.6 in [BBD].)

Example 6.1.4 ([BBD] 1.3.16)**.** If $\mathbf{D}'$ is a full triangulated subcategory of a t-category $\mathbf{D}$, then $\mathbf{D}'$ is also a t-category with the t-structure given by $(\mathbf{D}'^{\geq 0}, \mathbf{D}'^{\leq 0})$, where $\mathbf{D}'^{\geq 0} \overset{\text{def}}{=} \mathbf{D}^{\geq 0} \cap \mathbf{D}$ and $\mathbf{D}'^{\leq 0} \overset{\text{def}}{=} \mathbf{D}^{\leq 0} \cap \mathbf{D}'$.

Definition 6.1.5. Let $\varphi : \mathbf{D} \longrightarrow \mathbf{D}'$ be a triangulated functor between t-categories. We say that φ is *right t-exact* if $\varphi(\mathbf{D}^{\leq 0}) \subseteq \mathbf{D}'^{\leq 0}$, and *left t-exact* if $\varphi(\mathbf{D}^{\geq 0}) \subseteq \mathbf{D}'^{\geq 0}$. We say that φ is *t-exact* if it is both right and left t-exact.

The concept of (left or right) t-exactness is a generalization of exactness in abelian category. We have the following result regarding t-exact functors and the induced functor on the hearts.

Proposition 6.1.6 ([BBD] 1.3.17)**.** *Let $\mathbf{D}$ and $\mathbf{D}'$ be t-categories with hearts $\mathscr{A}$ and $\mathscr{A}'$ respectively. Furthermore, let $F : \mathbf{D} \longrightarrow \mathbf{D}'$ be a left (resp., right) t-exact triangulated functor. Then $\mathbf{H}^0 F$ is a left (resp., right) exact functor from $\mathscr{A}$ to $\mathscr{A}'$.*

If a t-category $\mathbf{D}$ is equipped with an additive symmetric monoidal structure that is right t-exact in both factors, then so is its heart $\mathscr{C}$. The symmetric monoidal structure on the heart is defined as follows. Suppose $- \otimes -$ is the tensor operator on $\mathbf{D}$. For C, C' in $\mathscr{C}$, we define $C \otimes^{\mathscr{C}} C'$ by $\mathbf{H}^0(C \otimes C')$. Since $\otimes$ is right t-exact in both factors, for all M and N in $\mathbf{D}^{\leq 0}$,

$$\mathbf{H}^0(M \otimes N) = \mathbf{H}^0(\mathbf{H}^0(M) \otimes \mathbf{H}^0(N))$$

([Dég10, 5.10]) and $\otimes^{\mathscr{C}}$ is well-defined. It is now straightforward to verify that $(\mathscr{C}, \otimes^{\mathscr{C}})$ satisfies all the axioms of a symmetric monoidal category.

In addition, if $\mathbf{D}$ has a partial internal hom structure $(\underline{\mathrm{Hom}}, \mathbf{D}^{\mathrm{rep}})$ as defined in Definition 4.2.2, then $\mathscr{C}$ is also equipped with a partial internal hom. For C, C' in $\mathscr{C}$, let us set

$$\underline{\mathrm{Hom}}_{\mathscr{C}}(C, C') \overset{\text{def}}{=} \mathbf{H}^0(\underline{\mathrm{Hom}}(C, C')).$$

Proposition 6.1.7. *Let C be an object in $\mathbf{D}^{\mathrm{rep}} \cap \mathscr{C}$ such that $\underline{\mathrm{Hom}}(C, -)$ is right t-exact. Then $\underline{\mathrm{Hom}}_{\mathscr{C}}(C, -)$ is right adjoint to $C \otimes^{\mathscr{C}} -$ as endofunctors on $\mathscr{C}$.*

Proof. Notice that for M in $\mathbf{D}^{\leq 0}$ and M' in $\mathbf{D}^{\geq 0}$,

$$\mathrm{Hom}_{\mathbf{D}}(\mathbf{H}^0(M), M') \cong \mathrm{Hom}_{\mathbf{D}}(M, M') \tag{6.1.8}$$

and

$$\mathrm{Hom}_{\mathbf{D}}(M, \mathbf{H}^0(M')) \cong \mathrm{Hom}_{\mathbf{D}}(M, M'). \tag{6.1.9}$$

Fix any C_1, C_2 in $\mathscr{C}$. Since $\otimes$ is right t-exact in both factors, $C_1 \otimes C$ is in $\mathbf{D}^{\leq 0}$, and using the isomorphism in (6.1.8), we obtain the following isomorphism:

$$\mathrm{Hom}_{\mathscr{C}}(C_1 \otimes^{\mathscr{C}} C, C_2) = \mathrm{Hom}_{\mathscr{C}}(\mathbf{H}^0(C_1 \otimes C), C_2) \cong \mathrm{Hom}_{\mathbf{D}}(C_1 \otimes C, C_3).$$

Since C is in $\mathbf{D}^{\mathrm{rep}}$, the functor $- \otimes C$ is left adjoint to $\underline{\mathrm{Hom}}(C, -)$. By assumption, $\underline{\mathrm{Hom}}(C, -)$ is right t-exact, and since C_2 is in the heart, $\underline{\mathrm{Hom}}(C, C_2)$ is an object in $\mathbf{D}^{\geq 0}$. Therefore, using the isomorphism in (6.1.9), we obtain the following chain of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(C_1 \otimes^{\mathscr{C}} C, C_2) &\cong \mathrm{Hom}_{\mathbf{D}}(C_1 \otimes C, C_2) \\
&\cong \mathrm{Hom}_{\mathbf{D}}(C_1, \underline{\mathrm{Hom}}(C, C_2)) \\
&\cong \mathrm{Hom}_{\mathbf{D}}(C_1, \mathbf{H}^0(\underline{\mathrm{Hom}}(C, C_2))) \\
&= \mathrm{Hom}_{\mathscr{C}}(C_1, \underline{\mathrm{Hom}}_{\mathscr{C}}(C, C_2)).
\end{aligned}$$

Since each of the above isomorphism is natural in C_1 and C_2, the proposition now follows. □

Proposition 6.1.7 shows that the functor $\underline{\mathrm{Hom}}_{\mathscr{C}}$ defines a partial internal hom on the category $\mathscr{C}$ where the collection of semi-representable objects of $\underline{\mathrm{Hom}}_{\mathscr{C}}$ contains at least those objects C in $\mathscr{C} \cap \mathbf{D}^{\mathrm{rep}}$ such that $\underline{\mathrm{Hom}}(C, -)$ is left t-exact.

The above discussion applies to the category $\mathbf{DM}^{\mathrm{eff},-}$ since there exists a t-structure on $\mathbf{DM}^{\mathrm{eff},-}$, with the abelian category $\mathbf{HI}$ as its heart (see Theorem 4.1.8).

Definition 6.1.10. We write $\tau_{\leq 0}\mathbf{DM}^{\mathrm{eff},-}$ for the negative objects of $\mathbf{DM}^{\mathrm{eff},-}$, and $\tau_{\geq 0}\mathbf{DM}^{\mathrm{eff},-}$ for the positive objects of $\mathbf{DM}^{\mathrm{eff},-}$. We will also let $\tau_{\leq 0}$, $\tau_{\geq 0}$ and $\mathbf{H}^0$ denote the functors from $\mathbf{DM}^{\mathrm{eff},-}$ to $\tau_{\leq 0}\mathbf{DM}^{\mathrm{eff},-}$, $\tau_{\geq 0}\mathbf{DM}^{\mathrm{eff},-}$ and $\mathbf{HI}$, respectively.

The triangulated monoidal structure on $\mathbf{DM}^{\mathrm{eff},-}$ induces a symmetric monoidal bifunctor on $\mathbf{HI}$, which we write as $\otimes^H$. This bifunctor is uniquely characterized by

$$h_X \otimes^H h_Y = h_{X \times Y},$$

where $h_X = \mathbf{H}^0(M(X))$ for X in Sm_k (see [Dég10, 1.8]).

Moreover, there is a partial internal hom, defined by

$$\underline{\mathrm{Hom}}_{\mathbf{HI}}(F, G) = \mathbf{H}^0(\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(F, G))$$

for all G in $\mathbf{HI}$, and F in $\mathbf{HI} \cap \mathbf{DM}^{\mathrm{eff},-}_{gm}$ for which $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(F, -)$ is left t-exact. Our first goal is to show that $\mathcal{O}^*$ is semi-representable with respect to $\underline{\mathrm{Hom}}_{\mathbf{HI}}$ by showing that $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathcal{O}^*, -)$ is right t-exact. This is established by the following lemma. Recall from Definition 3.2.7 that for F in $\mathbf{HI}$, F_{-1} is the contraction of F, which is the homotopy invariant sheaf with transfers defined by

$$X \mapsto \mathrm{cok}(F(X \times \mathbb{A}^1) \longrightarrow F(X \times (\mathbb{A}^1 - 0))).$$

Moreover, $F \mapsto F_{-1}$ defines an endofunctor on $\mathbf{HI}$.

Lemma 6.1.11. *There exists a natural isomorphism of homotopy invariant sheaves with transfers*

$$\mathbf{H}^i \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n)[n], M) \cong (\mathbf{H}^i M)_{-n}.$$

In particular, $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathcal{O}^*, -)$ *is left t-exact.*

Proof. Fix an object M in $\mathbf{DM}^{\mathrm{eff},-}$, regarded as a cochain complex of sheaves with transfers with homotopy invariant cohomology presheaves. By [Dég08, 3.4.4], there exists a natural morphism between homotopy invariant presheaves with transfers:

$$i : (H^i M)_{-n} \longrightarrow H^i \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n)[n], M)$$

such that for all fields E over k, the following is an isomorphism:

$$(H^i M)_{-n}(\mathrm{Spec}\, E) \cong H^i \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n)[n], M)(\mathrm{Spec}\, E).$$

By [MVW, 11.2], i induces a natural isomorphism of the associated homotopy invariant Nisnevich sheaves with transfers:

$$(\mathbf{H}^i M)_{-n} \xrightarrow{\cong} \mathbf{H}^i \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n)[n], M).$$

This proves the first claim in the Lemma.

To see that $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathcal{O}^*, -)$ is left t-exact, suppose M is a positive object, i.e., $\mathbf{H}^i M = 0$ for all $i < 0$. Since $\mathcal{O}^* \cong \mathbb{Z}(1)[1]$, applying the above for $n = 1$, we see that for all $i < 0$,

$$\mathbf{H}^i \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathcal{O}^*, M) \cong \mathbf{H}^i \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(1)[1], M) \cong (\mathbf{H}^i M)_{-1} = 0.$$

Thus, $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathcal{O}^*, M)$ is also a positive object in $\mathbf{DM}^{\mathrm{eff},-}$, and the lemma is now established. □

Definition 6.1.12. To emphasize the relationship with corresponding operations in $\mathbf{DM}^{\mathrm{eff},-}$, let us set

$$F(1)^{\mathrm{HI}} \stackrel{\mathrm{def}}{=} F \otimes^H \mathcal{O}^* \quad \text{and} \quad F_{-1}^{\mathrm{HI}} \stackrel{\mathrm{def}}{=} \underline{\mathrm{Hom}}_{\mathbf{HI}}(\mathcal{O}^*, F).$$

We write $F(n)^{\mathrm{HI}}$ for $(F(n-1)^{\mathrm{HI}})(1)^{\mathrm{HI}}$ and F_{-n}^{HI} for $(F_{-n+1}^{\mathrm{HI}})_{-1}^{\mathrm{HI}}$.

By Lemma 6.1.11 and preceding comments, $F \mapsto F(1)^{\mathrm{HI}}$ is left adjoint to $F \mapsto F_{-1}^{\mathrm{HI}}$, and therefore $F \mapsto F(n)^{\mathrm{HI}}$ is left adjoint to $F \mapsto F_{-n}^{\mathrm{HI}}$ for all $n > 0$.

Remark 6.1.13. To simplify notation, we will drop the "HI", and simply write $F(n)$ and F_{-n} for $F(n)^{\mathrm{HI}}$ and F_{-n}^{HI}. Doing so introduces a number of potential sources of ambiguity. The first is that $F(n)$ is already used to represent $F \otimes^L \mathbb{Z}(n)$, where $\mathbb{Z}(n)$ is the motivic complex in $\mathbf{DM}^{\mathrm{eff},-}$ introduced in Section 4.3. In particular, $\mathbb{Z}(n)$ may refer to the motivic complexes as well as the objects $\mathbb{Z} \otimes^H (\mathcal{O}^*)^{\otimes n}$. To resolve this ambiguity, we adopt the following convention: For the remainder of the thesis, unless otherwise specified, for an object F in $\mathbf{HI}$, $F(n)$ will denote $F(n)^{\mathrm{HI}} \stackrel{\mathrm{def}}{=} F \otimes^H (\mathcal{O}^*)^{\otimes n}$. All mentions of $\mathbb{Z}(n)$ will refer to the motivic complex in $\mathbf{DM}^{\mathrm{eff},-}$.

The second source of potential ambiguity comes from the fact that F_{-1} is already used to represent the contraction of the sheaf F in $\mathbf{HI}$. Recall from Definition 3.2.7 that F_{-1} is the sheaf that sends X in Sm_k to $\operatorname{cok} p^*$, where

$$p^* : F(X) \longrightarrow F(X \times (\mathbb{A}^1 - 0))$$

is the map induced by the projection $X \times (\mathbb{A}^1 - 0) \longrightarrow X$. In fact, there is no ambiguity here, since the contraction of F is isomorphic to the sheaf $\underline{\mathrm{Hom}}_{\mathbf{HI}}(\mathcal{O}^*, F)$. Indeed, by

[Dég08, 3.4.5], the contraction of F is isomorphic to $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(1)[1], F)$. Recall from [MVW, 4.1] that $\mathbb{Z}(1)[1] \cong \mathcal{O}^*$ in $\mathbf{DM}^{\mathrm{eff},-}$. Hence, we have that

$$\underline{\mathrm{Hom}}_{\mathbf{HI}}(\mathcal{O}^*, F) \cong \mathbf{H}^0\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(1)[1], F) \cong \mathbf{H}^0 F_{-1} = F_{-1}.$$

Finally, we make some observations that will be useful in subsequent sections.

Proposition 6.1.14. *For all negative objects M,*

$$\mathbf{H}^0(M \otimes^L \mathbb{Z}(n)[n]) = \mathbf{H}^0(M)(n).$$

Proof. By construction, the tensor operation $\otimes^L$ is right t-exact in both factors. Therefore, for negative objects M and N of $\mathbf{DM}^{\mathrm{eff},-}$, we have that

$$\mathbf{H}^0 M \otimes^H \mathbf{H}^0 N = \mathbf{H}^0(\mathbf{H}^0(M) \otimes^L \mathbf{H}^0(N)) = \mathbf{H}^0(M \otimes^L N).$$

Since $\mathbb{Z}(n)[n] = \mathbb{Z}(n-1)[n-1] \otimes^L \mathbb{Z}(1)[1]$, and $\mathbb{Z}(1)[1] \cong \mathcal{O}^*$, by induction on n, $\mathbb{Z}(n)[n]$ is also a negative object and $\mathbf{H}^0(\mathbb{Z}(n)[n]) \cong (\mathcal{O}^*)^{\otimes n}$. Moreover, for a negative object M in $\mathbf{DM}^{\mathrm{eff},-}$, we obtain the following:

$$\mathbf{H}^0(M \otimes^L \mathbb{Z}(n)[n]) \cong \mathbf{H}^0(M) \otimes^H (\mathcal{O}^*)^{\otimes n} = \mathbf{H}^0(M)(n).$$ □

Proposition 6.1.15. *Let F be a homotopy invariant sheaf with transfers. The unit map $F \longrightarrow F(n)_{-n}$ is an isomorphism.*

Proof. For F in $\mathbf{HI}$, by the Cancellation Theorem 4.4.1, we have that

$$\underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n)[n], F(n)[n]) \cong \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}, F) \cong F.$$

Now apply $\mathbf{H}^0$ to this chain of isomorphisms. Using Lemma 6.1.11 and the fact that $\mathbf{H}^0(F) = F$, we obtain the desired isomorphism. □

Proposition 6.1.16. *If $F = G(n)$ for some G in $\mathbf{HI}$, then $\epsilon_F^n : F_{-n}(n) \longrightarrow F$ is an isomorphism.*

Proof. Suppose $F = G(n)$ for some G in $\mathbf{HI}$. Writing L for the functor $F \mapsto F(n)$, by counit-unit adjunction, the composition

$$G(n) \xrightarrow{L\eta_G} (G(n)_{-n})(n) \xrightarrow{\epsilon_G L} G(n)$$

is the identity, where η_G and ϵ_G are the unit and the counit maps respectively. By Proposition 6.1.15, η_G is an isomorphism, and so is $L\eta_G$. It follows that $\epsilon_G L$ is an isomorphism as well. Since $\epsilon_G L$ is the counit map for $LG = G(n) = F$, the proposition follows. □

6.2 Torsion filtration on HI

We now define the first filtration on **HI**. Let $\mathbf{HI}(0) = \mathbf{HI}$ and let $\mathbf{HI}(n)$ denote the full subcategory of objects F where $F \cong F'(n)$ for some F' in **HI**. It is clear that if $m \geq n$, then $\mathbf{HI}(m) \subseteq \mathbf{HI}(n)$. In particular, we have a tower of subcategories

$$\mathbf{HI} = \mathbf{HI}(0) \supset \mathbf{HI}(1) \supset \mathbf{HI}(2) \subset \cdots .$$

To see that this filtration is not trivial (i.e., $\mathbf{HI}(n) \neq \mathbf{HI}(m)$ for all natural numbers $n \neq m$), notice that for the constant sheaf $\mathbb{Z}$, it is clear that $\mathbb{Z}_{-1} = 0$. Then $\mathbb{Z}$ is an object in **HI** but not in $\mathbf{HI}(1)$. Indeed, if $\mathbb{Z} \in \mathbf{HI}(1)$ then $\mathbb{Z} \cong F'(1)$, but then $\mathbb{Z}_{-1} = F'$ by Proposition 6.1.15, forcing $\mathbb{Z} = 0$. Similarly, since $\mathcal{O}^*_{-1} = \mathbb{Z}$, $\mathcal{O}^* \in \mathbf{HI}(1)$ but $\mathcal{O}^* \notin \mathbf{HI}(2)$. In general, $\mathcal{O}^*(n-1)$ is an object of $\mathbf{HI}(n)$ but not $\mathbf{HI}(n+1)$.

Remark 6.2.1. The subcategories $\mathbf{HI}(n)$ are additive, but *not* abelian, except for the case $n = 0$. To see this, consider the map

$$n : \mathcal{O}^* \longrightarrow \mathcal{O}^*$$

given by sending $u \in \mathcal{O}^*(X)$ to u^n for each X in Sm_k. The kernel of this map is the sheaf of n-th roots of unity μ_n. But $(\mu_n)_{-1} = 0$. If μ_n were in $\mathbf{HI}(n)$, then by Proposition 6.1.16, we would have $\mu_n \cong (\mu_n)_{-1}(1) = 0$, which is a contradiction. It follows that $\mathbf{HI}(1)$ is not closed under kernels. Similar arguments show that $\mathbf{HI}(n)$ is not closed under kernel for any positive integer n.

Recall from Definition 5.0.1 that a descending weak filtration $(\mathscr{A}_*, \varphi_*)$ is a tower of subcategories $\mathscr{A}_i$ together with coreflection functors $\varphi_i : \mathscr{A} \longrightarrow \mathscr{A}_i$ such that φ_i restricted to $\mathscr{A}_i$ is naturally isomorphic to the identity. To show that the full subcategories $\mathbf{HI}(n)$ define a descending weak filtration, we need to show that there exist coreflection functors $\sigma^n : \mathbf{HI} \longrightarrow \mathbf{HI}(n)$.

Definition 6.2.2. Let σ^n denote the functor $F \mapsto (F_{-n})(n)$. Since $F \mapsto F(n)$ is right exact, and F_{-n} is exact (Proposition 3.2.8), σ^n is right exact. However, σ^n is *not* always left exact (see Example 6.2.4 below).

Proposition 6.2.3. *The functor σ^n is right adjoint to the inclusion of $\mathbf{HI}(n)$. In particular, $(\mathbf{HI}(*), \sigma^*)$ defines a (nontrivial) descending weak filtration of $\mathbf{HI}$.*

Proof. Let $f : F \longrightarrow G$ be a map in $\mathbf{HI}$, with F in $\mathbf{HI}(n)$, and let ϵ^n denote the counit $\sigma^n \longrightarrow \mathrm{id}$. By naturality of ϵ^n, we have the following commutative diagram:

$$\begin{array}{ccc} F_{-n}(n) & \xrightarrow{\epsilon^n f} & G_{-n}(n) \\ \downarrow{\scriptstyle \epsilon^n_F} & & \downarrow{\scriptstyle \epsilon^n_G} \\ F & \xrightarrow{\quad f \quad} & G. \end{array}$$

Since $F \in \mathbf{HI}(n)$, by Proposition 6.1.16 the counit map ϵ^n_F is an isomorphism.

Define the map $\chi : \mathrm{Hom}_{\mathbf{HI}}(F, G) \longrightarrow \mathrm{Hom}_{\mathbf{HI}(n)}(F, G_{-n}(n))$ by $f \mapsto \epsilon^n f \circ (\epsilon^n_F)^{-1}$. Since $\epsilon^n_G \circ \chi(f) = f$, χ is injective. Moreover, given a map $g : F \longrightarrow G_{-n}(n)$, set $f' = \epsilon^n_G \circ g$. Then $\chi(f') = g$. Hence χ is an isomorphism as desired. From the way χ is defined, it is clear that χ is functorial in both F and G, and therefore σ^n is right adjoint to the inclusion of $\mathbf{HI}(n)$ into $\mathbf{HI}$.

To show that $(\mathbf{HI}(*), \sigma^*)$ define a weak descending filtration, the only criterion left to check is that σ^n restricted to $\mathbf{HI}(n)$ is naturally isomorphic to the identity. By Proposition 6.1.16, the counit map $\epsilon^n : \sigma^n F \longrightarrow F$ is an isomorphism for all F in $\mathbf{HI}(n)$, and the proposition follows. □

Example 6.2.4. While $(\mathbf{HI}(*), \sigma^*)$ forms a weak filtration of $\mathbf{HI}$, for a given sheaf F in $\mathbf{HI}$, the objects $\sigma^n F$ are not in general subobjects of F, because the counit map $\sigma^n F \longrightarrow F$ is not always injective. Here is an example.

Let $\mathcal{O}^{*n}$ be the sheaf of n-th power of global units associated to the presheaf where sections of a smooth finite type k-scheme X is the abelian subgroup of $\mathcal{O}^*$ given by

$$\mathcal{O}^{*n}(X) = \{x : x = y^n \text{ for some } y \text{ in } \mathcal{O}^*(X)\}.$$

It is clear that $\mathcal{O}^{*n} \in \mathbf{HI}$. Furthermore, there exists the following exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{O}^{*n} \longrightarrow 0$$

where μ_n is the constant sheaf of n-th roots of unity. In particular, $(\mu_n)_{-1} = 0$. By Proposition 3.2.8, the functor $F \mapsto F_{-1}$ is exact. Therefore, the map $\mathcal{O}^*_{-1} \longrightarrow (\mathcal{O}^{*n})_{-1}$ is an isomorphism, and

$$(\mathcal{O}^{*n})_{-1}(1) \cong \mathcal{O}^*_{-1}(1) = \mathcal{O}^*,$$

and the counit $(\mathcal{O}^{*n})_{-1}(1) \longrightarrow \mathcal{O}^{*n}$ is given precisely by $x \mapsto x^n$, which has a nontrivial kernel.

We can understand the problem in another way, which is that the categories $\mathbf{HI}(*)$ are too small and do not include all the kernels of counits $(F_{-n})(n) \longrightarrow F$. This can be fixed by enlarging the filtration at each level, and to do so, we turn to torsion theory.

Motivated by Example 6.2.4, we introduce the following more stringent criteria on weak filtrations.

Definition 6.2.5. Let $\mathscr{A}$ be an abelian category. We say that a $\mathbb{Z}$-indexed descending weak filtration $(\mathscr{A}_*, \varphi_*)$ is a *strong filtration* if for each A in $\mathscr{A}$ and n in $\mathbb{Z}$, $\varphi_n A \longrightarrow A$ is a monomorphism of A. An ascending weak filtration $(\mathscr{A}_*, \varphi_*)$ is a *strong cofiltration* if $A \longrightarrow \varphi_n A$ is a quotient of A for each n and each A in $\mathscr{A}$.

Similarly, we can define ascending strong filtration and descending strong cofiltration on $\mathscr{A}$.

Example 6.2.6. Here is an example of a strong ascending filtration and a strong descending filtration on the category $\mathbf{QC}$ of quasi-coherent sheaves on $\mathbb{P}^n$. Let i_k denote the closed immersion of $\mathbb{P}^k$ into $\mathbb{P}^n$ as a subscheme identified by the vanishing of the last $n-k$ homogeneous coordinates, and let j_k denote the open immersion of $U_k \stackrel{\text{def}}{=} \mathbb{P}^n - \mathbb{P}^k$ into $\mathbb{P}^n$.

Let $\mathbf{QC}^k$ be the full subcategory of quasi-coherent sheaves on $\mathbb{P}^n$ supported in U_k and let $\mathbf{QC}_k$ denote the full subcategory of sheaves on $\mathbb{P}^n$ supported in $\mathbb{P}^k$. Since

$$U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n$$

and

$$\mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq \cdots \subseteq \mathbb{P}^n$$

we have the following towers of subcategories:

$$\mathbf{QC}^0 \supseteq \mathbf{QC}^1 \supseteq \mathbf{QC}^2 \supseteq \cdots \supseteq \mathbf{QC}^n$$

and

$$\mathbf{QC}_0 \subseteq \mathbf{QC}_1 \subseteq \mathbf{QC}_2 \subseteq \cdots \subseteq \mathbf{QC}_n.$$

We will show that the towers of subcategories define a strong filtration and strong cofiltration on $\mathbf{QC}$. For each positive integer k less than n and each F in $\mathbf{QC}$, we have the following exact sequence of quasi-coherent sheaves on $\mathbb{P}^n$:

$$0 \longrightarrow (j_k)_!(F|_U) \longrightarrow F \longrightarrow (i_k)_*(F|_Z) \longrightarrow 0 \tag{6.2.7}$$

where $(j_k)_!(F|_U)$ is the sheaf associated with the presheaf given by

$$V \mapsto \begin{cases} F(V) & \text{if } V \subseteq U_k \\ 0 & \text{otherwise.} \end{cases}$$

In this case $(j_k)_!(F)$ is in $\mathbf{QC}^k$ and $(i_k)_*(F)$ is in $\mathbf{QC}_k$ (see [Hart77, Ex. 1.19]). In fact, $F \mapsto (j_k)_!(F|_{U_k})$ and $F \mapsto (i_k)_*(F|_{\mathbb{P}^k})$ define functors from $\mathbf{QC}$ to $\mathbf{QC}^k$ and $\mathbf{QC}_k$ respectively. In this case $(j_k)_!$ is right adjoint to inclusion, and $(i_k)_*$ is left adjoint to inclusion.

In general, let $Z_1 \subseteq Z_2 \subseteq \cdots Z_n$ be a sequence of subschemes of some scheme X, and let $U_k = X - Z_k$. Let $\mathbf{QC}(X)$ be the abelian category of quasi-coherent sheaves on X. Then there exists a strong descending filtration

$$\mathbf{QC}^0(X) \supseteq \mathbf{QC}^1(X) \supseteq \mathbf{QC}^2(X) \supseteq \cdots \supseteq \mathbf{QC}^n(X) \tag{6.2.8}$$

where $\mathbf{QC}^k(X)$ is the full subcategory of quasi-coherent sheaves supported on U_k, and a strong ascending cofiltration on $\mathbf{QC}(X)$

$$\mathbf{QC}_0(X) \subseteq \mathbf{QC}_1(X) \subseteq \mathbf{QC}_2(X) \subseteq \cdots \subseteq \mathbf{QC}_n(X) \tag{6.2.9}$$

where $\mathbf{QC}_k(X)$ is the full subcategory of quasi-coherent sheaves on X supported on Z_k. As above, the coreflection functors φ^k from $\mathbf{QC}$ to $\mathbf{QC}^k$ are given by $F \mapsto (j_k)_!(F|U_k)$ where j_k is the open immersion $j_k : U_k \longrightarrow X$; the reflection functors φ_k from $\mathbf{QC}$

to $\mathbf{QC}_k$ are given by $F \mapsto (i_k)_*(F|_{Z_k})$, where $i_k : Z_k \longrightarrow X$ is the evident closed immersion. Since, for each k, we have an exact sequence of quasi-coherent sheaves as in (6.2.7), $\varphi^k(F)$ is a subobject of F and $\varphi_k(F)$ is a quotient of F for each F in $\mathbf{QC}(X)$. The claim that (6.2.8) defines a strong filtration and (6.2.9) defines a strong cofiltration now follows.

We will now state the main theorem. Recall from Theorem 2.2.9 that if φ is a coradical, the associated torsion theory is a pair of full subcategories $(\mathscr{T}, \mathscr{F})$ where the torsion subcategory $\mathscr{T}$ consists of the objects T such that $\varphi(T) = 0$, and the torsionfree subcategory $\mathscr{F}$ consists of the objects F such that the map $F \longrightarrow \varphi(F)$ is an isomorphism.

Theorem 6.2.10. *There exists a sequence of coradicals $\varphi^{<n}, n = 0, 1, 2 \cdots$ on* $\mathbf{HI}$ *such that the associated torsionfree subcategories $\mathbf{HI}^{\geq *}$ form a descending strong filtration of* $\mathbf{HI}$ *and the associated torsion subcategories $\mathbf{HI}^{<*}$ form a strong cofiltration.*

Theorem 6.2.10 will be verified by Propositions 6.2.17 and 6.2.22 below. We first define the strong cofiltration, and show that the reflection functors are coradicals.

Definition 6.2.11. If n is a natural number, let $\mathbf{HI}^{<n}$ be the full subcategory of objects F in $\mathbf{HI}$ such that $F_{-n} = 0$. By Proposition 3.2.8, $F \mapsto F_{-n}$ is exact. Therefore, $\mathbf{HI}^{<n}$ is an abelian subcategory closed under extensions.

By convention, define F_{-0} to be F. Since $F_{-n-1} = (F_{-n})_{-1}$, we obtain the following ascending tower of subcategories

$$0 = \mathbf{HI}^{<0} \subset \mathbf{HI}^{<1} \subset \mathbf{HI}^{<2} \subset \cdots \subset \mathbf{HI}$$

Since $\mathcal{O}^*_{-1} = \mathbb{Z}$ and $\mathbb{Z}_{-1} = 0$, $\mathcal{O}^*$ is in $\mathbf{HI}^{<2}$ but not in $\mathbf{HI}^{<1}$. By Proposition 6.1.15 and by induction, $\mathcal{O}^*(n)$ is in $\mathbf{HI}^{<n+1}$ but not in $\mathbf{HI}^{<n}$.

We now describe the reflection functors $\varphi^{<n} : \mathbf{HI} \longrightarrow \mathbf{HI}^{<n}$.

Definition 6.2.12. Let n be a positive integer, and let $\varphi^{<n}(F)$ denote the cokernel of the counit $\epsilon^n_F : F_{-n}(n) \longrightarrow F$. Since ϵ^n_F is natural in F, $\varphi^{<n}$ is a functor.

We will show that $\varphi^{<n}$ is the desired reflection functor from **HI** to $\mathbf{HI}^{<n}$. This is established in Proposition 6.2.15. We proceed by first considering the following lemmas:

Lemma 6.2.13. *The image of* **HI** *under* $\varphi^{<n}$ *is contained in* $\mathbf{HI}^{<n}$*.*

Proof. Let F be an object of **HI**. We need to verify that $\varphi^{<n}(F)_{-n} = 0$. By definition, we have an exact sequence

$$F_{-n}(n) \longrightarrow F \longrightarrow \varphi^{<n}(F) \longrightarrow 0.$$

Since the functor $F \mapsto F_{-n}$ is exact (see Proposition 3.2.8 and Remark 6.1.13), we then have the following exact sequence

$$F_{-n}(n)_{-n} \longrightarrow F_{-n} \longrightarrow \varphi^{<n}(F)_{-n} \longrightarrow 0.$$

By Proposition 6.1.15, $(F_{-n}(n))_{-n} \longrightarrow F_{-n}$ is an isomorphism. Hence, $\varphi^{<n}(F)_{-n} = 0$ as desired. $\square$

Lemma 6.2.14. *The functor* $\varphi^{<n}$*, restricted to* $\mathbf{HI}^{<n}$ *is naturally isomorphic to the identity. Consequently, the functor* $\varphi^{<n}$ *is idempotent (see Definition 2.1.3 (2)), and the image of* **HI** *under* $\varphi^{<n}$ *is* $\mathbf{HI}^{<n}$*.*

Proof. For each F in $\mathbf{HI}^{<n}$, we have the following exact sequence:

$$F_{-n}(n) \longrightarrow F \longrightarrow \varphi^{<n}(F) \longrightarrow 0$$

Since $F \in \mathbf{HI}^{<n}$, $F_{-n} = 0$, and therefore the counit map is 0. It follows that the natural map $F \longrightarrow \varphi^{<n}(F)$ is a natural isomorphism as desired. The first statement follows from the fact that $\varphi^{<n}(F)$ is in $\mathbf{HI}^{<n}$, which is established in Lemma 6.2.13. $\square$

Proposition 6.2.15. *For each* n*, the functor* $\varphi^{<n}$ *is left adjoint to the inclusion of* $\mathbf{HI}^{<n}$ *into* **HI***.*

Proof. Let F be a homotopy invariant sheaf with transfers, and let G be an object in $\mathbf{HI}^{<n}$. For all $f : F \longrightarrow G$ we have the following commutative diagram:

$$\begin{array}{ccc}
F & \xrightarrow{\pi_F} & \varphi^{<n}(F) \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \varphi^{<n}(f)} \\
G & \xrightarrow{\pi_G} & \varphi^{<n}(G)
\end{array}$$

where π_F and π_G are surjections. By Lemma 6.2.14, the map $G \xrightarrow{\pi_G} \varphi^{<n}(G)$ is an isomorphism. Define

$$\chi : \mathrm{Hom}_{\mathbf{HI}}(F, G) \longrightarrow \mathrm{Hom}_{\mathbf{HI}^{<n}}(\varphi^{<n}(F), G)$$

by $f \mapsto \pi_G^{-1} \circ \varphi^{<n}(f)$. Since $\chi(f) \circ \pi_F = f$, χ is injective. For $g : \varphi^{<n}(F) \longrightarrow G$, set $f' = \pi \circ g$. Since $\chi(f') = g$, χ is a bijection, as desired.

From the way χ is defined, it is clear that χ is functorial in both F and G. The proposition now follows. □

This shows that $\varphi^{<n}$ is an idempotent quotient functor for each natural number n. In fact, we have the following result:

Proposition 6.2.16. *For each natural number n, $\varphi^{<n}$ is a coradical.*

Proof. By Lemma 6.2.14, $\varphi^{<n}$ is idempotent. By Proposition 6.2.15, $\varphi^{<n}$ is a left adjoint, and since $\mathbf{HI}^{<n}$ is an abelian category (see Definition 6.2.11), $\varphi^{<n}$ is therefore right exact. All that remains to show is that for each F in $\mathbf{HI}$,

$$\varphi^{<n}(\ker(F \longrightarrow \varphi^{<n}(F))) = 0.$$

Fix a positive integer n, and let K denote the kernel of the surjection $F \longrightarrow \varphi^{<n}(F)$. Since $\varphi^{<n}(F)$ is in $\mathbf{HI}^{<n}$, by definition $\varphi^{<n}(F)_{-n} = 0$. Therefore, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \sigma^n K & \longrightarrow & \sigma^n F & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_F} & & \downarrow{\scriptstyle \epsilon_F} & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & \varphi^{<n}(F) & \longrightarrow & 0.
\end{array}$$

By the Snake Lemma, and using the fact that cok $\epsilon_F = \varphi^{<n}(F)$, we have the exact sequence

$$0 \longrightarrow \varphi^{<n}(K) \longrightarrow \varphi^{<n}(F) \xrightarrow{q} \varphi^{<n}(F) \longrightarrow 0.$$

And the map q is the identity. It follows that $\varphi^{<n}(K) = 0$ as desired. □

Since $\varphi^{<n}$ is a coradical, by Theorem 2.2.6, there exists a torsion theory $(\mathscr{T}_n, \mathscr{F}_n)$ associated with each $\varphi^{<n}$. We now give another description of the torsionfree subcategories.

Proposition 6.2.17. *For each positive integer n, the full subcategory $\mathbf{HI}^{<n}$ and the torsionfree subcategory $\mathscr{F}_n$ are the same. Hence, the torsionfree subcategories form an ascending strong cofiltration of $\mathbf{HI}$.*

Proof. Recall from Lemma 6.2.13 that $\varphi^{<n}(F)_{-n} = 0$ for all F in $\mathbf{HI}$. Hence, if F is in $\mathscr{F}_n$, $F_{-n} = \varphi^{<n}(F)_{-n} = 0$.

Conversely, if $F_{-n} = 0$, then $\varphi^{<n}(F) = F$ by Lemma 6.2.14, and by definition F is an object of $\mathbf{HI}^{<n}$. Hence, the torsionfree subcategory $\mathscr{F}_n$ is precisely the full subcategory $\mathbf{HI}^{<n}$ of the sheaves F in $\mathbf{HI}$ for which $F_{-n} = 0$. This proves the first claim in the proposition. Since $\mathbf{HI}^{<n}$ form an ascending strong cofiltration, the second claim now follows. □

We still have to show that the torsion subcategories $\mathscr{T}_n$ form a strong descending filtration. Let us first introduce a more appropriate notation for the torsion subcategory.

Definition 6.2.18. Let $\mathbf{HI}^{\geq n}$ denote the torsion subcategory $\mathscr{T}_n$, and $\varphi^{\geq n}$ denote the kernel of the natural surjection $\mathrm{id} \longrightarrow \varphi^{<n}$. By Proposition 2.1.8 and Corollary 2.2.7, $\varphi^{\geq n}$ is an idempotent pre-radical, and is right adjoint to the inclusion of $\mathbf{HI}^{\geq n}$ in $\mathbf{HI}$.

We will now show that $(\mathbf{HI}^{\geq *}, \varphi^{\geq *})$ defines a descending strong filtration on $\mathbf{HI}$.

Lemma 6.2.19. *The essential image of $\varphi^{\geq n}$ is $\mathbf{HI}^{\geq n}$, and the restriction of $\varphi^{\geq n}$ to $\mathbf{HI}^{\geq n}$ is the identity.*

Proof. Recall from the definition of $\varphi^{\geq n}$ that for each F in $\mathbf{HI}$, there exists a short exact sequence:

$$0 \longrightarrow \varphi^{\geq n}F \longrightarrow F \longrightarrow \varphi^{<n}F \longrightarrow 0.$$

Furthermore, recall from Theorem 2.2.6 that the for all F in $\mathbf{HI}^{\geq n}$, $\varphi^{<n}F = 0$. The lemma now follows. □

Lemma 6.2.20. *For natural numbers n and m such that $m > n$, $\varphi^{<m}\varphi^{<n} = \varphi^{<n}$ and there exist a natural isomorphism $\varphi^{<n}\varphi^{<m} \cong \varphi^{<n}$.*

Proof. Suppose F is in $\mathbf{HI}$. Since $\mathbf{HI}^{<n}$ is a full subcategory of $\mathbf{HI}^{<m}$, and $\varphi^{<m}$ is the identity on $\mathbf{HI}^{<m}$ (Lemma 6.2.14), we have $\varphi^{<m}\varphi^{<n} = \varphi^{<n}$. It remains for us to show that $\varphi^{<n}\varphi^{<m} \cong \varphi^{<n}$.

We have the following commutative diagram:

$$\begin{array}{ccccccc} \sigma^n\sigma^m(F) & \longrightarrow & \sigma^n(F) & \longrightarrow & \sigma^n\varphi^{<m}(F) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \epsilon_{\sigma^m(F)}} & & \downarrow{\scriptstyle \epsilon_F} & & \downarrow{\scriptstyle \epsilon_{\varphi^{<m}(F)}} & & \\ \sigma^m(F) & \longrightarrow & F & \longrightarrow & \varphi^{<m}(F) & \longrightarrow & 0, \end{array} \tag{6.2.21}$$

where the vertical arrows are the counits. Furthermore, by the same arguments as in the Snake Lemma, we have the "snake tail" exact sequence:

$$\operatorname{cok} \epsilon_{\sigma^m(F)} \longrightarrow \varphi^{<n}(F) \longrightarrow \varphi^{<n}\varphi^{<m}(F) \longrightarrow 0.$$

However, since $\sigma^m F \in \mathbf{HI}(m)$, by Proposition 6.1.15 $\epsilon_{\sigma^m(F)}$ is an isomorphism. Therefore, the natural map $\varphi^{<n}(F) \xrightarrow{\cong} \varphi^{<n}\varphi^{<m}(F)$ is an isomorphism. □

Proposition 6.2.22. *The collection* $(\mathbf{HI}^{\geq *}, \varphi^{\geq *})$ *form a descending strong filtration of* $\mathbf{HI}$*, i.e., we have the following descending tower of subcategories*

$$\mathbf{HI} = \mathbf{HI}^{\geq 0} \supseteq \mathbf{HI}^{\geq 1} \supseteq \cdots \supseteq \mathbf{HI}^{\geq n} \supseteq \mathbf{HI}^{\geq n+1} \supseteq \cdots$$

and coreflection functors $\varphi^{\geq n} : \mathbf{HI} \longrightarrow \mathbf{HI}^{\geq n}$ *such that* $\varphi^{\geq n}$ *restricted to* $\mathbf{HI}^{\geq n}$ *is the identity, and* $\varphi^{\geq n}(F)$ *is a subobject of* F *for all* n.

Proof. The only claim left to show is that $\mathbf{HI}^{\geq m} \subseteq \mathbf{HI}^{\geq n}$ for $n \leq m$.

Let F be an object in $\mathbf{HI}^{\geq m}$. Then $\varphi^{<m}(F) = 0$, and by Lemma 6.2.20

$$0 = \varphi^{<n}\varphi^{<m}(F) = \varphi^{<n}(F).$$

Thus, F is in $\mathbf{HI}^{\geq n}$. □

We introduce the following notion to describe the strong filtration and cofiltration on $\mathbf{HI}$ and its relationship to the coradicals $\varphi^{<*}$.

Definition 6.2.23. We call the strong filtration and cofiltration defined by the torsion theories $(\mathbf{HI}^{\geq n}, \mathbf{HI}^{<n})$ for $n = 0, 1, 2, \ldots$ the *torsion filtration of* $\mathbf{HI}$.

In general, if $\mathscr{A}$ is an abelian category, we say that $\mathscr{A}$ has a torsion filtration if there exists a sequence of idempotent pre-(co)radicals $\varphi^{<*}$ such that the induced torsion theories $(\mathscr{A}^{\geq n}, \mathscr{A}^{<n})$ (for n in $\mathbb{Z}$) fit together to form a descending strong filtration

$$\mathscr{A} \supseteq \cdots \supseteq \mathscr{A}^{\geq 0} \supseteq \mathscr{A}^{\geq 1} \supseteq \cdots \supseteq \mathscr{A}^{\geq n} \supseteq \cdots$$

and an ascending strong cofiltration

$$0 \subseteq \cdots \subseteq \mathscr{A}^{<0} \subseteq \mathscr{A}^{<1} \subseteq \cdots \subseteq \mathscr{A}^{<n} \subseteq \cdots .$$

We conclude this section by presenting some additional properties of the torsion subcategories and the functor $\varphi^{\geq n}$. Recall from Proposition 6.2.3 that $\sigma^n F = F_{-n}(n)$.

Proposition 6.2.24. *For all natural numbers m and n such that $m > n$,*

1. $\mathbf{HI}^{\geq n}$ *is the full subcategory of objects F for which the counit map $\sigma^n(F) \longrightarrow F$ is onto.*

2. $\mathbf{HI}(n)$ *is a proper full subcategory of* $\mathbf{HI}^{\geq n}$*.*

3. *there exists a natural isomorphism between $\varphi^{<n}\varphi^{\geq m}$ and $\varphi^{\geq m}\varphi^{<n}$. Furthermore, $\varphi^{\geq n}\varphi^{<m} = \varphi^{<m}\varphi^{\geq n} = 0$*

4. *there exists natural isomorphisms: $\varphi^{\geq n}\varphi^{\geq m} \cong \varphi^{\geq m}\varphi^{\geq n} \cong \varphi^{\geq m}$.*

Proof. *(1)* : For all F in $\mathbf{HI}$ and $n \geq 0$, we have the following exact sequence

$$\sigma^n(F) \longrightarrow F \longrightarrow \varphi^{<n}(F) \longrightarrow 0.$$

Therefore, $\varphi^{<n}(F) = 0$ if and only if $\sigma^n(F) \longrightarrow F$ is a surjection.

(2) : Let F be an object in $\mathbf{HI}(n)$. Then $F \cong F'(n)$ for some F' in $\mathbf{HI}$. By Proposition 6.1.16, the counit map $\sigma^n(F) \longrightarrow F$ is an isomorphism. By part (1), $F \in \mathbf{HI}^{\geq n}$.

(3) : Let F be a homotopy invariant sheaf with transfers. Since $\varphi^{\geq m}(F) \in \mathbf{HI}^{\geq n}$, $\varphi^{<n}\varphi^{\geq m}(F) = 0$ by definition. Furthermore, $\varphi^{\geq m}\varphi^{<n}(F) = 0$ since it is the kernel of $\varphi^{<m}\varphi^{<n}(F) \longrightarrow \varphi^{<n}(F)$ which is an isomorphism by Lemma 6.2.19.

To show that $\varphi^{<n}\varphi^{\geq m}$ is naturally isomorphic to $\varphi^{\geq m}\varphi^{<n}$, let us first consider the following diagram:

$$\begin{array}{ccccccccc} & & \sigma^m\varphi^{\geq n}(F) & \longrightarrow & \sigma^m(F) & \longrightarrow & \sigma^m\varphi^{<n}(F) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varphi^{\geq n}(F) & \longrightarrow & F & \longrightarrow & \varphi^{<n}(F) & \longrightarrow & 0 \end{array} \tag{6.2.25}$$

where vertical maps are the counits. Notice that the top row is exact on the right because σ^m is right exact. Since $m > n$, by Lemma 6.2.13, $\varphi^{<n}(F)$ is in $\mathbf{HI}^{<n}$, which is a subcategory of $\mathbf{HI}^{<m}$ by Proposition 6.2.17. Since G is in $\mathbf{HI}^{<m}$ if and only if $G_{-m} = 0$, it follows that $(\varphi^{<n}(F))_{-m} = 0$. Hence, $\sigma^m\varphi^{<n}(F) = 0$.

Applying the Snake Lemma to (6.2.25), we obtain the following exact sequence:

$$0 \longrightarrow \varphi^{<m}\varphi^{\geq n}(F) \longrightarrow \varphi^{<m}(F) \longrightarrow \varphi^{<m}\varphi^{<n}(F) \longrightarrow 0. \tag{6.2.26}$$

Now $\varphi^{<n}\varphi^{<m}(F) \cong \varphi^{<n}(F)$, and the composition $\varphi^{<m}(F) \longrightarrow \varphi^{<n}\varphi^{<m}(F) \longrightarrow \varphi^{<n}(F)$ is precisely the natural surjection associated to $\varphi^{<n}(F)$. It follows that

$$\varphi^{<m}\varphi^{\geq n}(F) \cong \varphi^{\geq n}\varphi^{<m}(F).$$

Since (6.2.25) is natural in F, the isomorphism is natural in F as well.

(4) : By Proposition 6.2.22, $\mathbf{HI}^{\geq m} \subseteq \mathbf{HI}^{\geq n}$. Since $\varphi^{\geq n}$ restricted to $\mathbf{HI}^{\geq n}$ is the identity by Lemma 6.2.19, $\varphi^{\geq n}\varphi^{\geq m} = \varphi^{\geq m}$.

To show that $\varphi^{\geq m}\varphi^{\geq n} \cong \varphi^{\geq m}$, notice that for a given F in $\mathbf{HI}$ and positive integer n, there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varphi^{\geq n}(F) & \longrightarrow & F & \longrightarrow & \varphi^{<n}(F) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta_{\varphi^{\geq n}(F)}} & & \downarrow{\scriptstyle \eta_F} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \varphi^{<m}\varphi^{\geq n}(F) & \longrightarrow & \varphi^{<m}(F) & \longrightarrow & \varphi^{<m}\varphi^{<n}(F) & \longrightarrow & 0, \end{array} \tag{6.2.27}$$

where η is the natural surjection $\mathrm{id} \longrightarrow \varphi^{<m}$, and the bottom row is precisely the short exact sequence (6.2.26). By Lemma 6.2.20, the map $\varphi^{<n}(F) \longrightarrow \varphi^{<n}\varphi^{<m}(F)$ is an isomorphism. Therefore, by the Snake Lemma, we have $\varphi^{\geq m}\varphi^{\geq n}(F) \cong \varphi^{\geq m}(F)$. Since (6.2.27) is natural in F, it follows that the isomorphism $\varphi^{\geq m}\varphi^{\geq n} \longrightarrow \varphi^{\geq m}$ is natural as well. $\square$

6.3 Slice Filtration on $\mathbf{DM}^{\text{eff},-}$ and Torsion Filtration on HI

In this section, we want to relate the filtrations on **HI** that we have developed with the slice filtration on $\mathbf{DM}^{\text{eff},-}$. Recall that the slice filtration structure on $\mathbf{DM}^{\text{eff},-}$ is associated with the weak filtration $(\mathbf{DM}^{\text{eff},-}_{\geq *}, \nu^{\geq *})$ and the weak cofiltration $(\mathbf{DM}^{\text{eff},-}_{<*}, \nu^{<*})$ (see Section 5). The main result that we will verify is Proposition 6.3.1. Recall from Definition 6.1.10 that $\tau_{\leq 0}\mathbf{DM}^{\text{eff},-}$ is the full subcategory of negative objects in $\mathbf{DM}^{\text{eff},-}$, i.e., the objects M in $\mathbf{DM}^{\text{eff},-}$ such that $\mathbf{H}^n M = 0$ for all $n > 0$.

Proposition 6.3.1. *For each positive integer n, the following diagram of functors commute, with surjective vertical arrows:*

$$\begin{array}{ccccc}
\mathbf{DM}^{\text{eff},-}_{\geq n} & \xleftarrow[\nu^{\geq n}]{} & \tau_{\leq 0}\mathbf{DM}^{\text{eff},-} & \xrightarrow{\nu^{<n}} & \mathbf{DM}^{\text{eff},-}_{<n} \\
\downarrow{\mathbf{H}^0} & & \downarrow{\mathbf{H}^0} & & \downarrow{\mathbf{H}^0} \\
\mathbf{HI}(n) & \xleftarrow[\sigma^n]{} & \mathbf{HI} & \xrightarrow{\varphi^{<n}} & \mathbf{HI}^{<n}.
\end{array}$$

The rest of the section will be devoted to the proof of Proposition 6.3.1. First, observe that for every positive integer n and every M in $\tau_{\leq 0}\mathbf{DM}^{\text{eff},-}$, there exists a slice triangle:

$$\nu^{\geq n}M \longrightarrow M \longrightarrow \nu^{<n}M \longrightarrow \nu^{\geq n}M[1]$$

Applying the cohomological functor $\mathbf{H}^0$, we obtain the following long exact sequence

$$\cdots \xrightarrow{\delta_{-1}} \mathbf{H}^0\nu^{\geq n}(M) \longrightarrow \mathbf{H}^0 M \longrightarrow \mathbf{H}^0\nu^{<n}(M) \xrightarrow{\delta_0} \mathbf{H}^1\nu^{\geq n}(M) \longrightarrow \cdots \qquad (6.3.2)$$

where $\mathbf{H}^i M \stackrel{\text{def}}{=} \mathbf{H}^0 M[i]$.

Since $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\text{eff},-}}(\mathbb{Z}(n)[n], -)$ is t-exact as shown in Lemma 6.1.11, the cochain complex $\underline{\mathrm{RHom}}_{\mathbf{DM}^{\text{eff},-}}(\mathbb{Z}(n)[n], M)$ is also in $\tau_{\leq 0}\mathbf{DM}^{\text{eff},-}$. By Proposition 6.1.14,

$$\begin{aligned}
\mathbf{H}^0\nu^{\geq n}M &= \mathbf{H}^0(\underline{\mathrm{RHom}}_{\mathbf{DM}^{\text{eff},-}}(\mathbb{Z}(n)[n], M) \otimes^L \mathbb{Z}(n)[n]) \\
&\cong \mathbf{H}^0(\underline{\mathrm{RHom}}_{\mathbf{DM}^{\text{eff},-}}(\mathbb{Z}(n)[n], M))(n) \\
&= \mathbf{H}^0(M)_{-n}(n) \\
&= \sigma^n \mathbf{H}^0 M.
\end{aligned}$$

This shows that the left square of Proposition 6.3.1 commutes, that $\mathbf{HI}(n)$ is equal to the image of $\tau_{\leq 0}\mathbf{DM}^{\mathrm{eff},-}_{\geq n}$ under $\mathbf{H}^0$, and the coreflection functors from $\tau_{\leq 0}\mathbf{DM}^{\mathrm{eff},-}$ to $\mathbf{HI}(n)$ is compatible with $\mathbf{H}^0$.

To prove the commutativity of the right square, notice that, for M in $\tau_{\leq 0}\mathbf{DM}^{\mathrm{eff},-}$, $\mathbf{H}^0\nu^{\geq n}M = (\mathbf{H}^0 M)_{-n}(n)$. Hence, we get the following exact sequence from (6.3.2)

$$((\mathbf{H}^0 M)_{-n})(n) \longrightarrow \mathbf{H}^0 M \longrightarrow \mathbf{H}^0\nu^{<n}(M) \xrightarrow{\delta_0} \mathbf{H}^1\nu^{\geq n}(M).$$

where the map $((\mathbf{H}^0 M)_{-n}(n) \longrightarrow \mathbf{H}^0 M$ is the counit. If we show that $\mathbf{H}^1\nu^{\geq n}(M) = 0$, then it is clear that $\mathbf{H}^0\nu^{<n}(M) \cong \varphi^{<n}(\mathbf{H}^0 M)$. This shows that the right square of Proposition 6.3.1 commutes, completing the proof of Proposition 6.3.1. The vanishing of $\mathbf{H}^1\nu^{\geq n}$ is established by the following lemma:

Lemma 6.3.3. *For all positive integers n and all M in $\tau_{\leq 0}\mathbf{DM}^{\mathrm{eff},-}$, $\mathbf{H}^1\nu^{\geq n}(M) = 0$.*

Proof. Since $\nu^{\geq n}M = \mathbb{Z}(n)[n] \otimes^L \underline{\mathrm{RHom}}_{\mathbf{DM}^{\mathrm{eff},-}}(\mathbb{Z}(n)[n], M)$, we have already shown in the preceding discussion that $\nu^{\geq n}M$ is a negative object. Therefore, $\mathbf{H}^1\nu^{\geq n}M = 0$, as desired. □

6.4 Fundamental Invariants of the Torsion Filtration

As in the case of $\mathbf{DM}^{\mathrm{eff},-}$, we can also define the structure invariants associated to the filtration and cofiltration. In this case, for every natural number n, there exists a functorial exact sequence

$$\sigma^n \longrightarrow \sigma^{n-1} \longrightarrow \varphi^{<n}\sigma^{n-1} \longrightarrow 0.$$

Definition 6.4.1. We define *n-th slice functor on* $\mathbf{HI}$ to be the functor $s^n \overset{\mathrm{def}}{=} \varphi^{<n+1}\sigma^n$.

Recall from Definition 5.2.13 that the n-th slice functor of $(\mathbf{DM}^{\mathrm{eff},-}_{\geq *}, \nu^{\geq *})$ is the triangulated endofunctor ν^* that fits into the following exact triangle

$$\nu^{<n+1} \longrightarrow \nu^{<n} \longrightarrow \nu^n \longrightarrow \nu^{<n+1}[1].$$

A consequence of Proposition 6.3.1 is that the slice functors s^n on $\mathbf{HI}$ agree with the slice functors ν^n on $\mathbf{DM}^{\mathrm{eff},-}$ in the following sense:

Corollary 6.4.2. *For all natural numbers n, the slice functors satisfy*

$$\mathbf{H}^0\nu^n = s^n.$$

Proof. Applying $\mathbf{H}^0$ to the functorial triangle from (5.2.12), we obtain the following functorial exact sequence in **HI**:

$$\mathbf{H}^0\nu^{\geq n+1} \longrightarrow \mathbf{H}^0\nu^{\geq n} \longrightarrow \mathbf{H}^0\nu^n \longrightarrow \mathbf{H}^1\nu^{\geq n+1}.$$

By Proposition 6.1.14, $\mathbf{H}^0\nu^{\geq n} = \sigma^n$ and $\mathbf{H}^0\nu^{\geq n+1} = \sigma^{n+1}$, and by Lemma 6.3.3, $\mathbf{H}^1\nu^{\geq n+1} = 0$. It follows that $\mathbf{H}^0\nu^n = s^n$ as desired. □

Let us first consider the following proposition:

Proposition 6.4.3. *For natural numbers m and n, $\sigma^n\varphi^{<m}$ is naturally isomorphic to $\varphi^{<m}\sigma^n$, and are both 0 if $m \leq n$.*

Proof. Let F be an object in **HI**, and write L for the functor $F \mapsto F(1)$ and R for the functor $F \mapsto F_{-1}$. Since $\sigma^n = L^n R^n$, by Lemma 5.2.5, we have the commutative square

$$\begin{array}{ccc} \sigma^m\sigma^n(F) & \xrightarrow{f} & \sigma^n(F) \\ \downarrow \cong & & \| \\ \sigma^n\sigma^m(F) & \xrightarrow{g} & \sigma^n(F), \end{array} \tag{6.4.4}$$

where f is the counit of $\sigma^m\sigma^n(F) \longrightarrow \sigma^n(F)$ and g is obtained by applying σ^n to the counit $\sigma^m(F) \longrightarrow F$. The cokernel of f is precisely $\varphi^{<m}\sigma^n(F)$. Since σ^n is right exact, and the following sequence is exact

$$\sigma^m(F) \longrightarrow F \longrightarrow \varphi^{<m}(F) \longrightarrow 0,$$

the following sequence is also exact.

$$\sigma^n\sigma^m(F) \longrightarrow \sigma^n(F) \longrightarrow \sigma^n\varphi^{<m}(F) \longrightarrow 0.$$

It follows that the cokernel of g is $\sigma^n\varphi^{<m}(F)$. By the Five Lemma 6.4.3, $\varphi^{<m}\sigma^n(F) \cong \sigma^n\varphi^{<m}(F)$. Since the square in Lemma 6.2.13 is functorial, it follows that the isomorphism identified above is natural in F.

Finally, suppose $m \leq n$. Then by Proposition 6.2.15 $\varphi^{<m}(F)_{-n} = 0$. It follows that $\sigma^n\varphi^{<m}(F) = 0$, and $\varphi^{<m}\sigma^n(F) = 0$ as well. □

Remark 6.4.5. In case the indexing becomes difficult to keep track, one might wish to consider a "bread" analogy. Imagine that a half-infinite loaf of bread is laid out on a line marked from 0 to ∞ (representing an F in **HI**), and one is allowed to take cuts at the marked points and subsequently pick up all the bread lying greater than n or less than n. For the functors $\varphi^{\geq n}$ and σ^n, the higher the n, the less bread one would *take*. For the functors $\varphi^{<n}$, the greater the n, the less bread one would *leave*.

If one finds the analogy useful, one might wish to interpret Lemma 6.2.20, and Propositions 6.2.24 (3) and (4) with this culinary picture in mind.

As we did for the filtration $(\mathbf{HI}(*), \sigma^*)$ in Definition 6.4.1, we can define the structure invariants for $(\mathbf{HI}^{\geq *}, \varphi^{\geq *})$.

Definition 6.4.6. For each F in **HI** and natural number n, write φ^n for the functor $\varphi^{<n+1}\varphi^{\geq n}$, which we define to be the *n-th fundamental invariant of F associated to $\varphi^{\geq *}$* or simply the *n-th fundamental invariant.*

As it turns out, the n-th fundamental invariant is *not* the same as the n-th slice functor on **HI**. To see this, consider the example introduced in Example 6.2.4. For $\mathcal{O}^{*n}$, from the discussion in *loc. cit.*, we have that

$$s^k(\mathcal{O}^{*n}) = s^k(\mathcal{O}^*) = \begin{cases} \mathcal{O}^* & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

However, a simple calculation reveals that

$$\varphi^k(\mathcal{O}^{*n}) = \begin{cases} \mathcal{O}^{*n} & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Nonetheless, the n-th slice functor is related to the n-th fundamental invariant via the following proposition:

Proposition 6.4.7. *Let m and n be natural numbers such that $m > n$. There exists a natural surjection from $\varphi^{<m}\sigma^n$ to $\varphi^{<m}\varphi^{\geq n}$. In particular, for each F in* **HI***, there exists a natural surjection $\pi_m : s^m F \longrightarrow \varphi^m F$.*

Proof. Let F be an object of **HI**. We have the following short exact sequence:

$$0 \longrightarrow \varphi^{\geq n}\varphi^{<m}(F) \longrightarrow \varphi^{<m}(F) \longrightarrow \varphi^{<n}\varphi^{<m}(F) \longrightarrow 0.$$

By Lemma 6.2.20, $\varphi^{<m}\varphi^{<n}(F) = \varphi^{<n}(F)$, and therefore $\varphi^{\geq n}\varphi^{<m}(F)$ is the kernel of the surjection $\varphi^{<m}(F) \longrightarrow \varphi^{<n}(F)$. But the sequence

$$\sigma^n\varphi^{<m}(F) \longrightarrow \varphi^{<m}(F) \longrightarrow \varphi^{<n}(F) \longrightarrow 0$$

is exact. Therefore, the induced map from $\sigma^n\varphi^{<m}(F)$ to $\varphi^{\geq n}\varphi^{<m}(F)$ is a surjection as well. Furthermore, since the commutative diagram

$$\begin{array}{ccccccccc} & & \sigma^n\varphi^{<m}(F) & \longrightarrow & \varphi^{<m}(F) & \longrightarrow & \varphi^{<n}(F) & \longrightarrow & 0 \\ & & \downarrow\!\!\!\downarrow & & \| & & \| & & \\ 0 & \longrightarrow & \varphi^{\geq n}\varphi^{<m}(F) & \longrightarrow & \varphi^{<m}(F) & \longrightarrow & \varphi^{<n}(F) & \longrightarrow & 0 \end{array}$$

is functorial in F, the surjection is natural. This establishes the first claim of the proposition, since $\varphi^{\geq n}\varphi^{<m}$ is naturally isomorphic to $\varphi^{<m}\varphi^{\geq n}$ (Proposition 6.2.24 (3)) and $\sigma^n\varphi^{<m}$ is naturally isomorphic to $\varphi^{<m}\sigma^n$ (Proposition 6.4.3). The second claim follows by setting $n = m - 1$. □

6.5 Weakly Filtered Monoidal Structure on HI

We end this chapter by discussing the tensor properties of the torsion filtration. Let us first consider the following notion:

Definition 6.5.1. Let $(\mathscr{C}, \otimes, \mathbb{1})$ be a monoidal category. We say that $\mathscr{C}$ is a *weakly filtered monoidal category* if there exists a weak filtration $(\mathscr{C}_*, \varphi_*)$ such that for all integers m and n and C in $\mathscr{C}_m$ and C' in $\mathscr{C}_n$, $C \otimes C'$ is in $\mathscr{C}_{n+m}$.

Example 6.5.2. Here are two examples of weakly filtered monoidal categories that we have encountered in this thesis. Recall from Definition 5.3.2 that $\mathbf{DM}_{\geq k}$ is the full subcategory of the objects (M, n) in $\mathbf{DM}$ such that $n \geq k$. For (M, n) in $\mathbf{DM}_{\geq k}$ and (M', n') in $\mathbf{DM}_{\geq l}$, $(M, n) \otimes^L (M', n')$ is equal to $(M \otimes M', n + n')$, which is an object in $\mathbf{DM}_{\geq k+l}$. Therefore, the triangulated tensor product on $\mathbf{DM}$ is weakly filtered by $(\mathbf{DM}_{\geq *}, \nu^{\geq *})$.

Similarly, $(\mathbf{HI}(*), \sigma^*)$ defines a graded symmetric monoidal category on $\mathbf{HI}$ under $\otimes^H$. To see this, recall from the first paragraph in Section 6.2 that F is in $\mathbf{HI}(n)$ if $F \cong F'(n)$. Furthermore, since $F(n) = F \otimes^H (\mathcal{O}^*)^{\otimes n}$, $F(n) \otimes^H G(m) = (F \otimes^H G)(n+m)$. Therefore, $\mathbf{HI}(n) \otimes^H \mathbf{HI}(m) \subseteq \mathbf{HI}(n+m)$.

We now will show that $(\mathbf{HI}^{\geq *}, \varphi^{\geq *})$ defines a weakly filtered monoidal category on $\mathbf{HI}$. We begin by proving the following proposition:

Proposition 6.5.3. *For F in $\mathbf{HI}^{\geq n}$ and G in $\mathbf{HI}^{\geq m}$, $F \otimes^H G$ is an object of $\mathbf{HI}^{\geq n+m}$.*

Proof. Since $\mathbf{HI}^{\geq n+m}$ is the torsion subcategory associated to the coradical $\varphi^{<n+m}$, to show that $F \otimes^H G$ is in $\mathbf{HI}^{\geq n+m}$, it suffices to show that $\varphi^{<n+m}(F \otimes^H G) = 0$. Since G is in $\mathbf{HI}^{\geq n}$, by Proposition 6.2.24(1), the counit $\epsilon : L^m R^m(G) \longrightarrow G$ is surjective. Since $\otimes^L$ is right t-exact in both factors, the functor $F \otimes^H -$ is right exact by Proposition 6.1.6, and the following map is surjective:

$$F \otimes^H L^m R^m(G) \xrightarrow{\epsilon_F \otimes^H G} F \otimes^H L^m R^m(G). \tag{6.5.4}$$

Similarly, we see that the following map is also surjective:

$$F \otimes^H L^m R^m(G) \xrightarrow{L^n R^n(F) \otimes^H \epsilon_G} F \otimes^H G. \tag{6.5.5}$$

Composing (6.5.4) and (6.5.5), we obtain a surjection

$$f : L^n R^n(F) \otimes^H L^m R^m(G) \longrightarrow F \otimes^H G.$$

On the other hand, since $L^n R^n(F) \otimes^H L^m R^m(G) = L^{n+m}(R^n(F) \otimes^H R^m(G))$, the object $L^n R^n(F) \otimes^H L^m R^m(G)$ is in $\mathbf{HI}(n+m)$, and by Proposition 6.2.24,

$$\varphi^{<n+m}(L^n R^n(F) \otimes^H L^m R^m(G)) = 0.$$

Since $\varphi^{<n+m}$ is a coradical, which is right exact, the map

$$\varphi^{<n+m}(f) : \varphi^{<n+m}(L^n R^n(F) \otimes^H L^m R^m(G)) \longrightarrow \varphi^{<n+m}(F \otimes^H G)$$

is onto. Therefore, $\varphi^{<n+m}(F \otimes^H G) = 0$ and $F \otimes^H G$ is an object in $\mathbf{HI}^{\geq n+m}$, as desired. $\square$

The following is a direct consequence of Proposition 6.5.3.

Corollary 6.5.6. *Let* $\otimes^H$ *be the tensor product on* $\mathbf{HI}$ *defined in Definition 6.1.10. The strong filtration* $(\mathbf{HI}^{\geq *}, \varphi^{\geq *})$ *makes* $(\mathbf{HI}, \otimes^H)$ *into a weakly filtered monoidal category.*

Chapter 7

Filtration on CycMod

In this chapter, we will extend the torsion filtration on **HI** to the Rost-Déglise category of homotopy modules $\mathbf{HI}_*$ (see Definition 7.1.1 below). To further simplify notation, in this chapter, let $L : \mathbf{HI} \longrightarrow \mathbf{HI}$ denote the functor $F \mapsto F(1)$, and let $R : \mathbf{HI} \longrightarrow \mathbf{HI}$ denote the functor given by $F \mapsto F_{-1}$. We write $\epsilon^n : \mathrm{id} \longrightarrow R^n L^n$ and $\eta^n : L^n R^n \longrightarrow \mathrm{id}$ for the unit and counit maps; we abbreviate η^1 as η, and ϵ^1 as ϵ. The extension of these filtrations to $\mathbf{HI}_*$ is new.

7.1 Torsion filtration on $\mathbf{HI}_*$

Recall from [Dég10, 1.17] the following definition:

Definition 7.1.1. A *homotopy module* is a $\mathbb{Z}$-graded homotopy invariant sheaf with transfers F_* such that for every n, there exists a map $s_n : F_n(1) \longrightarrow F_{n+1}$ such that the corresponding adjunction map $w_n : F_n \longrightarrow (F_{n+1})_{-1}$ is an isomorphism. We call s_n and w_n the *n-th suspension* and the *n-th delooping* respectively. A morphism $F_* \longrightarrow G_*$ between homotopy module is a sequence of morphisms $F_n \longrightarrow G_n$ of homotopy invariant sheaves with transfers that commute with s_n and w_n.

Let $\mathbf{HI}_*$ denote the category of *homotopy modules*. Objects in $\mathbf{HI}_*$ will be represented by (F_*, w_*) where F_* is the $\mathbb{Z}$-graded homotopy invariant sheaf with transfers, and w_* is the sequence of deloopings.

There is a fully faithful functor $\sigma^\infty : \mathbf{HI} \longrightarrow \mathbf{HI}_*$ given by $F \mapsto (F_*, w_*)$ where

$$F_n = \begin{cases} F(k) & \text{if } n > 0 \\ F & \text{if } n = 0 \\ F_{-|n|} & \text{otherwise,} \end{cases}$$

and the n-th delooping $F_n \longrightarrow (F_{n+1})_{-1}$ is the unit map for $n \geq 0$ and the tautological natural isomorphism for $n < 0$. Furthermore, σ^∞ has a right adjoint $\omega^\infty : \mathbf{HI}_* \longrightarrow \mathbf{HI}$ given by $(F_*, w_*) \mapsto F_0$ (see [Dég10, 1.18]). Since σ^∞ is fully faithful and admits a right adjoint, we can regard $\mathbf{HI}$ as a full coreflective subcategory of $\mathbf{HI}_*$. The torsion filtration on $\mathbf{HI}$, defined in Definitions 6.2.18 and 6.2.11, gives rise to two $\mathbb{N}$-indexed weak filtrations of $\mathbf{HI}_*$. The goal is to extend these filtrations to a $\mathbb{Z}$-indexed strong filtration and a $\mathbb{Z}$-indexed cofiltration of $\mathbf{HI}_*$. In particular, we show that there is a sequence of coradicals $\varphi_*^{<n}$ on $\mathbf{HI}_*$ such that for nonnegative n, the restriction of $\varphi_*^{<n}$ to $\mathbf{HI}$ is $\varphi^{<n}$. In this case, the associated torsion theories will extend the torsion filtrations on $\mathbf{HI}$ to $\mathbf{HI}_*$.

The following proposition will be crucial to extending the functors $\varphi^{<n}$:

Proposition 7.1.2. *For F in* **HI** *and all positive numbers k and n, there are natural isomorphisms:*

$$L^k\varphi^{<n}(F) \cong \varphi^{<n+k}L^k(F) \tag{7.1.3}$$

and

$$R^k\varphi^{<n}(F) \cong \varphi^{<n-k}R^k(F). \tag{7.1.4}$$

Proof. By Lemma 5.2.1, the following diagram, natural in F, is commutative:

$$\begin{array}{ccc} L^{n+1}R^{n+1}L(F) & \xrightarrow{\epsilon^{n+1}L} & L(F) \\ \downarrow & & \| \\ L(L^nR^n)(F) & \xrightarrow{L\epsilon^n} & L(F). \end{array} \tag{7.1.5}$$

Here, ϵ^n denotes the counit $L^nR^n \longrightarrow \mathrm{id}$, and the vertical map $L^{n+1}R^{n+1}L(F) \longrightarrow L(L^nR^n(F))$, which is given by the map $L^{n+1}R^n\eta^{-1}$, where η is the unit $\mathrm{id} \longrightarrow RL$, is an isomorphism by Proposition 6.1.15.

The cokernel of $\epsilon^{n+1}L$ is $\varphi^{<n+1}L(F)$. Since L is right exact, the cokernel of $L\epsilon^n$ is $L\varphi^{<n}(F)$. By the Five Lemma, it is clear that $\varphi^{<n+1}L(F) \cong L\varphi^{<n}(F)$. Since (7.1.5) is natural in F, the isomorphism $\varphi^{<n+1}L \longrightarrow L\varphi^{<n}$ is natural as well. By similar arguments, one can show that $R\varphi^{<n}$ is naturally isomorphic to $\varphi^{<n-1}R$ as well. This proves the proposition for the case $k = 1$. The general case follows by induction. $\square$

We will now define the coradicals on $\mathbf{HI}_*$.

Definition 7.1.6. Let (F_*, w_*) be an object of $\mathbf{HI}_*$, and write $\varphi_*^{<n}(F)$ for the graded homotopy invariant sheaf with transfers where

$$(\varphi_*^{<n}(F))_k \stackrel{\text{def}}{=} \begin{cases} \varphi^{<n+k}(F_k) & \text{if } n+k>0 \\ 0 & \text{otherwise.} \end{cases}$$

For ease of notation, we will write $\varphi_k^{<n}(F)$ for the k-th graded component of $\varphi_*^{<n}(F)$.

Using the isomorphism $R\varphi^{<n}(F_k) \cong \varphi^{<n-1}R(F_k)$ established in Proposition 7.1.2, let

$$\varphi_k^{<n}(w) : \varphi_{k-1}^{<n}(F) \longrightarrow R\varphi_k^{<n}(F)$$

denote the composition

$$\varphi^{<n+k-1}(F_{k-1}) \xrightarrow{\varphi^{<n+k-1}(w_k)} \varphi^{<n+k-1}R(F_k) \xrightarrow{\cong} R\varphi^{<n+k}(F_k), \tag{7.1.7}$$

where $w_k : F_{k-1} \longrightarrow R(F_k)$ is the k-th delooping of (F_*, w_*). Since w_k is an isomorphism for all k, so is $\varphi_k^{<n}(w)$. Defining $\varphi_k^{<n}(s)$ to be the adjoint of $\varphi_k^{<n}(w)$, we immediately have that $(\varphi_*^{<n}(F), \varphi_k^{<n}(w))$ is an object of $\mathbf{HI}_*$.

Remark. In the discussion above, the map $\varphi_k^{<n}(s) : L\varphi_k^{<n}(F) \longrightarrow \varphi_{k+1}^{<n}(F)$ is actually given by the composition

$$L\varphi^{<n+k}(F_k) \xrightarrow{\cong} \varphi^{<n+k+1}L(F_k) \xrightarrow{\varphi^{<n+k+1}(s_k)} \varphi^{<n+k+1}(F_{k+1}),$$

where $s_k : LF_k \longrightarrow F_{k+1}$ is k-th suspension map. This also follows from Proposition 7.1.2.

Lemma 7.1.8. *For each integer n, $\varphi_*^{<n}$ is an endofunctor of* $\mathbf{HI}_*$.

Proof. Let $f_* : (F_*, w_*) \longrightarrow (G_*, w'_*)$ be a map between homotopy modules, and let $\varphi_*^{<n}(f)$ be a map of graded homotopy invariant sheaves with transfers whose k-th graded component is $\varphi_k^{<n}(f) \stackrel{\text{def}}{=} \varphi^{<n+k}(f_k)$. If we can show that $\varphi_*^{<n}(f)$ is a map in $\mathbf{HI}_*$, then it will be clear that $\varphi_*^{<n}$ preserves identity maps and compositions.

By naturality of $\rho : R\varphi^{<n+1} \longrightarrow \varphi^{<n}R$ and $\lambda : L\varphi^{<n} \longrightarrow \varphi^{<n+1}L$ and also by the above arguments, the following two squares are commutative

$$\begin{array}{ccc}
R\varphi^{<n+k}(F_k) & \xrightarrow{\quad R\varphi^{<n+k}(f_k)\quad} & R\varphi^{<n+k}(G_k) \\
\Big\downarrow{\scriptstyle \varphi^{<n+k-1}R(w)\rho} & & \Big\downarrow{\scriptstyle \varphi^{<n+k-1}R(w')\rho} \\
\varphi^{<n+k-1}(F_{k-1}) & \xrightarrow{\quad \varphi^{<n+k-1}(f_{k-1})\quad} & \varphi^{<n+k-1}(G_{k-1})
\end{array}$$

$$\begin{array}{ccc}
L\varphi^{<n+k}(F_k) & \xrightarrow{\quad L\varphi^{<n+k}(f_k)\quad} & L\varphi^{<n+k}(G_k) \\
\Big\downarrow{\scriptstyle \varphi^{<n+k+1}L(s)\rho} & & \Big\downarrow{\scriptstyle \varphi^{<n+k+1}L(s')\rho} \\
\varphi^{<n+k+1}(F_{k+1}) & \xrightarrow{\quad \varphi^{<n+k+1}(f_{k-1})\quad} & \varphi^{<n+k+1}(G_{k+1}).
\end{array}$$

Here, $\varphi_*^{<n}(f)$ is a map from $\varphi_*^{<n}(F)$ to $\varphi_*^{<n}(G)$ as homotopy modules. The fact that $\varphi_*^{<n}$ respects composition follows from the functoriality of $\varphi^{<*}$. $\square$

We now verify the main result of this section:

Theorem 7.1.9. *For each integer n, $\varphi_*^{<n}$ is a coradical of* $\mathbf{HI}_*$.

Proof. $\varphi_*^{<n}$ is a quotient functor: certainly $F_* \longrightarrow \varphi_*^{<n}(F)$ is surjective for each n since it is a surjection at each degree. What we need to verify is that the degree-wise surjection gives rise to a map of homotopy modules. In particular, we need to verify that the following diagram is commutative

$$\begin{array}{ccc}
L(F_k) & \xrightarrow{\quad s\quad} & F_{k+1} \\
\Big\downarrow & & \Big\downarrow \\
L\varphi^{<n+k}(F_k) & \xrightarrow{\ s\ } & \varphi^{<n+k-1}(F_{k+1}).
\end{array}$$

To see this, notice that the above diagram is the outer square of the diagram:

$$\begin{array}{ccc}
L(F_k) & \xrightarrow{\quad s\quad} & F_{k+1} \\
\Big\downarrow & & \Big\downarrow \\
\varphi^{<n+k+1}L(F_k) & \longrightarrow & \varphi^{<n+k+1}F_{k+1} \\
\Big\downarrow{\scriptstyle \lambda^{-1}} & & \Big\| \\
L\varphi^{<n+k}(F_k) & \xrightarrow{\ s\ } & \varphi^{<n+k+1}F_{k+1}.
\end{array}$$

Here, the top square commutes by the naturality of id $\longrightarrow \varphi^{<n+k}$, and the bottom square commutes by the definition of the suspension map $s : L\varphi^{<n+k}(F_k) \longrightarrow \varphi^{<n+k+1}(F_{k+1})$.

The fact that $\varphi_*^{<n}$ respects delooping follows from the duality of the suspension and delooping as established by the preceding lemma.

$\varphi_*^{<n}$ *is a pre-coradical*: The kernel of $F_* \longrightarrow \varphi_*^{<n}(F)$ is a homotopy module K_* whose k-th graded term is

$$\ker(F_k \longrightarrow \varphi^{<n+k}(F_k)).$$

But $\varphi^{<n}$ is a coradical; hence, $\varphi_*^{<n}(K_*) = \varphi^{<n+k}(K_k) = 0$. That is $\varphi_*^{<n}(K) = 0$, as desired.

$\varphi_*^{<n}$ *is a right exact*: since $\varphi^{<n+k}$ is right exact for each k, $\varphi_k^{<n}$ is right exact for each associated graded term. It follows that $\varphi_*^{<n}$ is right exact. □

Recall from Theorem 2.2.6 that if φ is a coradical, then the torsion subcategory of φ is the full subcategory $\mathscr{T}$ consisting of the objects T such that $\varphi(T) = 0$, and the torsionfree subcategory of φ is the full subcategory $\mathscr{F}$ whose objects are the objects F such that $\varphi(F) = 0$. Furthermore, by Corollary 2.2.7, the inclusion of $\mathscr{F}$ into the ambient category admits a right adjoint given by the kernel of the natural surjection id $\longrightarrow \varphi$.

Definition 7.1.10. For each integer i, let $\mathbf{HI}_*^{\geq n}$ and $\mathbf{HI}_*^{<n}$ denote the torsion and torsionfree subcategory of $\varphi_*^{<n}$ respectively. Let $\varphi_*^{\geq n}$ denote the kernel of the natural surjection id $\longrightarrow \varphi_*^{<n}$. By the preceding remarks, $\varphi_*^{\geq n}$ is right adjoint to the inclusion $\mathbf{HI}_*^{\geq n}$ in $\mathbf{HI}_*$.

Here is a straightforward consequence of Theorem 2.2.6 and Theorem 7.1.9:

Corollary 7.1.11. *An object (F_*, w_*) is in $\mathbf{HI}_*^{\geq n}$ if and only if $\varphi_*^{\geq n}(F) = (F_*, w_*)$.*

We now verify the main result of this section.

Corollary 7.1.12. *There exists a $\mathbb{Z}$-indexed torsion filtration on $\mathbf{HI}_*$. That is, there exists a $\mathbb{Z}$-indexed sequence of coradicals $\varphi_*^{<i}$ such that the associated torsion subcategories, which are given by $\mathscr{F}_i = \mathbf{HI}_*^{<i}$ form an ascending strong cofiltration of $\mathbf{HI}_*$:*

$$\cdots \subseteq \mathbf{HI}_*^{<-1} \subseteq \mathbf{HI}_*^{<0} \subseteq \mathbf{HI}_*^{<1} \subseteq \cdots \subseteq \mathbf{HI}_*^{<i} \subseteq \mathbf{HI}_*^{<i+1} \subseteq \cdots \subseteq \mathbf{HI}_*$$

and the associated torsionfree subcategories $\mathscr{T}_i = \mathbf{HI}_*^{\geq i}$ form a descending strong filtration of $\mathbf{HI}_*$:

$$\cdots \subseteq \mathbf{HI}_*^{\geq i} \subseteq \mathbf{HI}_*^{\geq i-1} \subseteq \cdots \subseteq \mathbf{HI}_*^{\geq 1} \subseteq \mathbf{HI}_*^{\geq 0} \subseteq \mathbf{HI}_*^{\geq -1} \subseteq \cdots\cdots \subseteq \mathbf{HI}_*.$$

Proof. Since (F_*, w_*) is in $\mathbf{HI}_*$ if and only if $\varphi_*^{<n}(F) = (F_*, w_*)$ and $\varphi_*^{<n}$ is idempotent, the restriction of $\varphi_*^{<n}$ to $\mathbf{HI}_*^{<n}$ is therefore the identity. Similarly, the restriction of $\varphi_*^{\geq n}$ to $\mathbf{HI}_*^{\geq n}$ is the identity.

What remains to be checked are that $\mathbf{HI}_*^{<n} \subset \mathbf{HI}_*^{<n+1}$ and $\mathbf{HI}_*^{\geq n+1} \subset \mathbf{HI}_*^{\geq n}$ for each integer n. To proceed, notice that by Lemma 6.2.20, $\varphi^{<n+1}\varphi^{<n} = \varphi^{<n}$. Therefore, for (F_*, w_*) in $\mathbf{HI}_*^{<n}$, if $\varphi_*^{<n}(F) = (F_*, w_*)$ then for every k, $\varphi^{<n+k+1}(F_k) = \varphi^{<n+k+1}\varphi^{<n+k}(F_k) = \varphi^{<n+k}(F_k) = F_k$. It follows that $\varphi_k^{<n+1}(F) = F_k$ for all k, and (F_*, w_*) is in $\mathbf{HI}_*^{<n+1}$. Hence, $\mathbf{HI}_*^{<n+1} \subset \mathbf{HI}_*^{<n+1}$ for every n. Using Proposition 6.2.24(4) and Corollary 7.1.12, we can show that $\mathbf{HI}_*^{\geq n+1} \subset \mathbf{HI}_*^{\geq n}$ for every n by using similar arguments. □

Example 7.1.13. Suppose F is a homotopy invariant sheaf with transfers. The image of F under σ^∞ (see the paragraph after Definition 7.1.1) is the homotopy module F_* where

$$F_k \overset{\text{def}}{=} \begin{cases} F(k) & \text{if } k > 0 \\ F & \text{if } k = 0 \\ F_{-|k|} & \text{otherwise.} \end{cases}$$

By Proposition 7.1.2, there are natural isomorphisms

$$\varphi^{<n+k}(F(k)) \cong (\varphi^{<n}F)(k) \quad \text{and} \quad \varphi^{<n}(F_{-k}) \cong (\varphi^{<n+k}F)_{-k}.$$

If F is in $\mathbf{HI}^{<n}$, then $\varphi^{<n}F = F$ by Proposition 6.2.17, and therefore $\varphi_*^{<n}(F_*) \cong F_*$. Similarly, if F is in $\mathbf{HI}^{\geq n}$, then $\varphi_*^{\geq n}(F_*) \cong F_*$. It follows that the image of $\mathbf{HI}^{\geq n}$ under σ^∞ is $\mathbf{HI}_*^{\geq n}$ and $\mathbf{HI}^{<n}$ under σ^∞ is $\mathbf{HI}_*^{<n}$ for each positive integer n.

The following result relates the coradicals $\varphi^{<n}$ on **HI** defined in Definition 6.2.12 to the coradicals $\varphi_*^{<n}$ on $\mathbf{HI}_*$. Recall the assumption from the opening paragraph of this chapter that L is the functor $F \mapsto F(1)$, and R is the functor $F \mapsto F_{-1}$.

Proposition 7.1.14. *For all integers $n \leq 0$, $\varphi_*^{<n}\sigma^\infty = 0$. Moreover, for each positive integer n, there exists a natural isomorphism $\sigma^\infty\varphi^{<n} \xrightarrow{\cong} \varphi_*^{<n}\sigma^\infty$.*

Proof. Let F be a homotopy invariant sheaf with transfers, and let $F_* = \sigma^\infty(F)$. According to the definition of σ^∞ in the paragraph after Definition 7.1.1,

$$F_k = \begin{cases} L^k F & \text{if } k > 0 \\ F & \text{if } k = 0 \\ R^{|k|}F & \text{if } k < 0. \end{cases}$$

First, suppose n is nonpositive. As defined in Definition 7.1.6, $\varphi_k^{<n}(F) = 0$ for all $k \leq 0$. Therefore, it suffices to show $\varphi_k^{<n}(F) = \varphi^{<n+k} = 0$ for all $k > 0$. Notice that $F_k = L^k F$, which is an object in $\mathbf{HI}^{\geq k}$ by Proposition 6.2.24(2). Since $\mathbf{HI}^{\geq k}$ is the torsion subcategory of the coradical $\varphi^{<k}$, $\varphi^{<k}(L^k F) = 0$. On the other hand, since $n + k < k$, by Lemma 6.2.20, $\varphi_*^{<n+k}(F) = \varphi^{<n+k}\varphi^{<k}(L^k F) = 0$. This proves the first statement in the proposition.

To see that there exists a natural isomorphism $\sigma^\infty\varphi^{<n} \xrightarrow{\cong} \varphi_*^{<n}\sigma^\infty$, we need to show that there exists a sequence of natural isomorphism $\tau_k : \sigma^\infty(\varphi^{<n}(F))_k \longrightarrow \varphi_k^{<n}\sigma^\infty(F)$ that is compatible with the delooping maps. That is, for each integer k, the following diagram commutes:

$$\begin{array}{ccc} R\varphi_{k+1}^{<n}(\sigma^\infty(F)) & \xrightarrow{\varphi_*^{<n}w_k} & \varphi_k^{<n}(\sigma^\infty(F)) \\ \downarrow{\scriptstyle R\tau_{k+1}} & & \downarrow{\scriptstyle \tau_k} \\ R\sigma^\infty(\varphi^{<k}(F))_{k+1} & \xrightarrow{w_k} & \sigma^\infty(\varphi^{<k})_k. \end{array}$$

Notice that the k-th graded component of $\sigma^\infty(\varphi^{<n}(F))$ is given by

$$\sigma^\infty(\varphi^{<n}(F))_k = \begin{cases} L^k\varphi^{<n}F & \text{if } k > 0 \\ \varphi^{<n}F & \text{if } k = 0 \\ R^{|k|}\varphi^{<n}F & \text{if } k < 0, \end{cases}$$

and by definition of $\varphi_*^{<n}$,

$$\varphi_k^{<n}(F) = \begin{cases} \varphi^{<n+k} L^k F & \text{if } k > 0 \\ \varphi^{<n} F & \text{if } k = 0 \\ \varphi^{<n+k} R^{|k|} F & \text{otherwise.} \end{cases}$$

To complete the proof of the proposition, we need to show that for each $k \geq 0$, there exists a natural isomorphism $\tau_k : L^k \varphi^{<n} \xrightarrow{\cong} \varphi^{<n+k} L^k$ such that the following diagram commutes:

$$\begin{array}{ccc} RL^{k+1}\varphi^{<n} & \xrightarrow{w_k} & L^k\varphi^{<n} \\ \Big\downarrow {\scriptstyle R\tau_{k+1}} & & \Big\downarrow {\scriptstyle \tau_k} \\ R\varphi^{<n+k+1}L^{k+1} & \xrightarrow{w'_k} & \varphi^{<n+k}L^k. \end{array} \tag{7.1.15}$$

Here, w_k is the natural isomorphism given by the inverse of the unit $\mathrm{id} \longrightarrow RL$ and w'_k is the natural isomorphism corresponding to the k-th deloopings of $\varphi_*^{<n}\sigma^\infty$. We must also show that for each $k < 0$, there exists a natural isomorphism $\tau_k : L\varphi^{<n+k}R^{|k|} \longrightarrow LR^{|k|}\varphi^{<n}$ such that the following diagram commutes:

$$\begin{array}{ccc} R^{|k|}\varphi^{<n} & = & R^{|k|}\varphi^{<n} \\ \Big\downarrow {\scriptstyle R\tau_{k+1}} & & \Big\downarrow {\scriptstyle \tau_k} \\ R\varphi^{<n+k+1}R^{|k+1|} & \xrightarrow{w'_k} & \varphi^{<n+k}R^{|k|} \end{array} \tag{7.1.16}$$

where w'_k is the natural isomorphism corresponding to the k-th delooping of $\varphi_*^{<n}\sigma^\infty$.

We proceed by first defining the natural isomorphisms τ_k. For $k \geq 0$, let τ_k be the natural isomorphism $L^k\varphi^{<n} \longrightarrow \varphi^{<n+k}L^k$ given by (7.1.3) of Proposition 7.1.2; for $k < 0$, let τ_k be the natural isomorphism $R^{|k|}\varphi^{<n} \longrightarrow \varphi^{<n+k}R^{|k|}$ given by (7.1.4). To see that (7.1.15) is commutative for all $k \geq 0$, notice that the inductive step in the proof of Proposition 7.1.2 shows that $R\tau_{k+1}$ factors as the composition

$$RL^{k+1}\varphi^{<k} \xrightarrow{RL\tau_k} RL\varphi^{<n+k}L^k \xrightarrow{R\tau'} R\varphi^{<n+k+1}L^{k+1},$$

where τ' is the natural isomorphism given by (7.1.3) for the case $k = 1$. In particular, the following diagram is commutative:

$$\begin{array}{ccc} RLL^k\varphi^{<n} & = & RL^{k+1}\varphi^{<n} \\ \downarrow{\scriptstyle R\tau_{k+1}} & & \downarrow{\scriptstyle RL\tau_k} \\ R\varphi^{<n+k+1}L^{k+1} & \xrightarrow{(R\tau')^{-1}} & RL\varphi^{<n+k}L^k. \end{array} \tag{7.1.17}$$

By the naturality of the inverse of the unit transformation, we also have the following commutative square:

$$\begin{array}{ccc} RL^{k+1}\varphi^{<n} & \xrightarrow{w_k} & L^k\varphi^{<n} \\ \downarrow{\scriptstyle RL\tau_k} & & \downarrow{\scriptstyle \tau_k} \\ RL\varphi^{<n+k}L^k & \xrightarrow{\eta\varphi^{<n+k}L^k} & \varphi^{<n+k}L^k. \end{array} \tag{7.1.18}$$

Furthermore, notice that w_k factors as

$$R\varphi^{<n+k+1}L^{k+1} \xrightarrow{(R\tau')^{-1}} RL\varphi^{<n+k}L^k \xrightarrow{\eta\varphi^{<n+k}L^k} \varphi^{<n+k}L^k.$$

The commutative squares (7.1.17) and (7.1.18) fit together to give us the square in (7.1.15). We have shown that (7.1.15) is commutative. By similar arguments, (7.1.16) is also commutative. Therefore, the natural isomorphisms w_k define a natural isomorphism $\sigma^\infty\varphi^{<n} \cong \varphi_*^{<n}\sigma^\infty$, which proves the second statement of the proposition. □

7.2 Torsion filtration on cycle modules

We conclude this chapter by showing that there is a torsion filtration structure on the category of cycle modules (defined below). Recall from [Mil70] that for a field F, the Milnor K-theory of F is the graded commutative ring given by

$$K_*^M(F) \stackrel{\text{def}}{=} T^*(F^*)/I$$

where $T^*(F^*)$ denotes the tensor algebra of the multiplicative group F^*, and I denotes the ideal generated by $a \otimes (1-a)$ for all a in F^*. We define $K_n^M(F)$ to be 0 for $n < 0$ and let $K_n^M(F)$ be the n-th graded piece of $K_*^M(F)$. We call $K_n^M(F)$ the *n-th Milnor K-theory of F*.

Definition 7.2.1 ([Ro96] 1.1)**.** Let X be a finite-type k-scheme, and let $\mathcal{F}(k)$ be the category of function fields E of Sm_k, i.e., E is the function field of some k-scheme X

in Sm_k, and any morphism $E \longrightarrow E'$ in $\mathcal{F}(k)$ is a field homomorphism such that the restriction to k is the identity. A *cycle premodule* M is a functor which assigns to every field E in $\mathcal{F}(k)$ a $\mathbb{Z}$-graded abelian group $M(E) = \{M_i\}_{i\in\mathbb{Z}}$, together with the following data:

D1. For each field extension $\varphi : E' \longrightarrow E$, there is a degree 0 map $\varphi_* : M(E') \longrightarrow M(E)$ called the *restriction map associated to* φ

D2. For each finite extension $\varphi : E' \longrightarrow E$, there is a degree 0 map $\varphi^* : M(E) \longrightarrow M(E')$ called the *corestriction map associated to* φ

D3. For each E in $\mathcal{F}(k)$, the group $M(E)$ is equipped with the structure of a left $K_*^M(E)$-module, where $K_*^M(E)$ is the Milnor K-ring of E.

D4. For a given valuation v of E in $\mathcal{F}(k)$, there exists a map of degree -1 $\partial_v : M(E) \longrightarrow M(\kappa(v))$ called the *residue* map, where $\kappa(v)$ is the residue field of v.

The data given in D1 - D4 satisfy the following criteria. For a given valuation v of E in $\mathcal{F}(k)$, fix p to be a prime of v. The $K_*^M(E)$-module structure in D3 and the residue map in D4 give rise to a degree preserving map $s_v^p : M(E) \longrightarrow M(\kappa(v))$ defined by $s_v^p(\rho) = \partial_v(\{p\}\cdot\rho)$, where $\{p\}$ the element in $K_1^M(E)$ represented by E. Following [Ro96, 1.1], we call s_v^p the *specialization* map.

R1a. For each field extension $\varphi : E' \longrightarrow E$ and field extension $\psi : E \longrightarrow E''$, $(\psi\circ\varphi)_* = \psi_* \circ \varphi_*$

R1b. For each finite extension $\varphi : E' \longrightarrow E$ and finite extension $\psi : E \longrightarrow E''$, $(\psi\circ\varphi)^* = \varphi^* \circ \psi^*$

R1c. For finite extension $\varphi : E' \longrightarrow E$ and any field extension $\psi : E' \longrightarrow E''$ with φ finite, define $R = E \otimes_{E'} E''$, and let $\mathfrak{p}$ be any prime ideal of R. (As R is Artin, let l_p be the length of the local ring $R_{(\mathfrak{p})}$), and $\varphi_{\mathfrak{p}} : E'' \longrightarrow R/\mathfrak{p}$ and $\psi_{\mathfrak{p}} : E \longrightarrow R/\mathfrak{p}$ be natural maps.

$$\psi_* \circ \varphi^* = \sum_{\mathfrak{p}} l_p \cdot (\varphi_{\mathfrak{p}})^* \circ (\psi_{\mathfrak{p}})_*.$$

R2. For any extension $\varphi : E' \longrightarrow E$, $x \in K_*^M(E')$, $y \in K_*^M(E)$, $\rho \in M(E')$, and $\mu \in M(E)$, then:

R2a. $\varphi_*(x \cdot \rho) = \varphi_*(x) \cdot \varphi_*(\rho)$.

R2b. if φ is finite, $\varphi^*(\varphi_*(x) \cdot \mu) = x \cdot \varphi^*(\mu)$.

R2c. if φ is finite, $\varphi^*(y \cdot \varphi_*(\rho)) = \varphi^*(y) \cdot \rho$.

R3. For any field extension $\varphi : E' \longrightarrow E$, v a valuation on E and w and a valuation on E':

R3a. Suppose w is a nontrivial restriction of v with ramification index e. Let $\overline{\varphi} : \kappa(w) \longrightarrow \kappa(v)$ be the induced map. Then:

$$\partial_v \circ \varphi_* = e \cdot \overline{\varphi}_* \circ \partial_w.$$

R3b. Let φ be a finite extension, suppose w is an extension of v to E. Let $\varphi_v : \kappa(w) \longrightarrow \kappa(v)$ be the induced map on the residue fields. Then

$$\partial_v \circ \varphi^* \sum_v \circ \partial_v.$$

R3c. Suppose v restricts to a trivial valuation on E'. Then

$$\partial_v \circ \varphi_* = 0$$

R3d. Suppose v restricts to a trivial valuation on E'. Let $\overline{\varphi} : F \longrightarrow \kappa(v)$ be the induced map on the residue fields. Let p a prime of v. Then

$$s_v^p \circ \varphi_* = \overline{\varphi}_*$$

R3e. Let u be an element of E such that $v(u) = 0$. Given ρ in $M(F)$, one has

$$\partial_v(\{u\} \cdot \rho) = -\{\overline{u}\} \cdot \partial_v(\rho).$$

For X a k-scheme, let $X^{(1)}$ denote the collection of codimension 1 subschemes. Let ξ_X be the generic point of an irreducible X with $K_X = \mathcal{O}_X, \xi_X$. If X is normal, then for x in $X^{(1)}$, the local ring $\mathcal{O}_{x,X}$ is a valuation ring of K_X with residue field $\kappa(x)$. Write $M(x)$ for $M(\kappa(x))$, and $\partial_x : M(\xi_X) \longrightarrow M(x)$ for the restriction map.

Furthermore, for $x, y \in X$, let Z be the closed subscheme determined by x, and $\overline{Z}$ be the normalization Z. Define

$$\partial_y^x : M(x) \longrightarrow M(y)$$

by

$$\partial_y^x = \begin{cases} 0 & y \notin Z^{(1)} \\ \sum_{z|y} \varphi^*_{\kappa(z),\kappa(x)} \circ \partial_z & \text{otherwise.} \end{cases}$$

Here, following [Ro96], $z|y$ denotes the relation that z lies over y. In particular, if $y \in Z^{(1)}$, the sum is taken over all z lying over $y \in Z^{(1)}$. In this case, $\varphi^*_{\kappa(z),\kappa(y)}$ is the corestriction map associated to the finite field extension $\kappa(y) \longrightarrow \kappa(z)$.

Definition 7.2.2 ([Ro96] 2.1)**.** A cycle module M on $\mathcal{F}(k)$ is a cycle premodule that satisfies the following conditions:

(FD) FINITE SUPPORT OF DIVISORS. X be a normal scheme and $\rho \in M(\xi_X)$. Then $\partial_x : M(\xi_X) \longrightarrow M(X)$ is 0 for all but finitely many $x \in X^{(1)}$.

(C) CLOSEDNESS. If X is an integral local scheme of dimension 2 with closed point x_0, then the map from $M(\xi_X)$ to $M(x_0)$ given by

$$\sum_{x \in X^{(1)}} \partial_x^{x_0} \circ \partial_\xi^x$$

is 0.

Déglise showed in [Dég10] that a homotopy module (F_*, w_*) gives rise to a unique cycle module $\widehat{F_*}$, and that this association defines an equivalence between the category of homotopy modules and cycle modules (see [Dég10, 3.7]). Via this categorical equivalence, we obtain the following corollary:

Corollary 7.2.3. *There exists a $\mathbb{Z}$-indexed torsion filtration on* **CycMod**. *That is, there exists a $\mathbb{Z}$-indexed sequence of coradicals, which by abuse of notation, we also* *represent by* $\varphi_*^{<i}$ *such that the associated torsion subcategories* $\mathbf{CycMod}^{<i}$ *form an ascending strong cofiltration of* $\mathbf{CycMod}$*:*

$$\cdots \subseteq \mathbf{CycMod}^{<-1} \subseteq \mathbf{CycMod}^{<0} \subseteq \cdots \subseteq \mathbf{CycMod}^{<i} \subseteq \cdots \subseteq \mathbf{CycMod}$$

and the associated torsionfree subcategories $\mathbf{CycMod}^{\geq i}$ *form a descending strong filtration of* $\mathbf{CycMod}$*:*

$$\cdots \subseteq \mathbf{CycMod}^{\geq i} \subseteq \cdots \subseteq \mathbf{CycMod}^{\geq 0} \subseteq \mathbf{CycMod}^{\geq -1} \subseteq \cdots\cdots \subseteq \mathbf{CycMod}.$$

Example 7.2.4. Milnor K-theory K_*^M, defined in the paragraph preceding Definition 7.2.1, is an example of a cycle module (see [Ro96, 1.4, 2.5]). By [Dég10, 3.7], the homotopy module corresponding to K_*^M is $\sigma^\infty(\mathbb{Z})$. As we have shown in Example 7.1.13, $\sigma^\infty(\mathbb{Z})$ is an object of $\mathbf{HI}_*^{\geq 1} \cap \mathbf{HI}_*^{<0}$. Hence, $K_*^M \in \mathbf{CycMod}^{\geq 1} \cap \mathbf{CycMod}^{<0}$.

Chapter 8

Torsion Filtrations on Torsion Monoidal Categories

In this chapter, we generalize the key results proven in the last three chapters by axiomatizing the necessary components to define torsion filtrations on t-categories $\mathbf{D}$ with a triangulated tensor structure. Let us begin by defining the following notion:

Definition 8.0.1. Let $(\mathbf{D}, \otimes, \mathbb{1})$ be a tensor monoidal category with a t-structure, and let $\mathscr{C}$ be its heart. We say that $\mathbf{D}$ is a *torsion monoidal category* if $\mathbf{D}$ is equipped with

1. (Partial Internal Hom) a partial internal hom structure $(\underline{\mathrm{Hom}}, \mathbf{D}^{\mathrm{rep}})$ (see Definition 4.2.1).
2. (Tate Object) an object S in both $\mathbf{D}^{\mathrm{rep}}$ and the heart of $\mathbf{D}$ called the *Tate object*. In particular, $\underline{\mathrm{Hom}}(S, -)$ is right adjoint to $S \otimes -$.

such that the following conditions hold:

1. $\mathbb{1}$ is an object of $\mathscr{C}$,
2. $\otimes$ is right t-exact in both factors,
3. (Cancellation) $\underline{\mathrm{Hom}}(S, S \otimes M) = M$,
4. $\underline{\mathrm{Hom}}(S, -)$ is t-exact.

If $\mathbf{D}$ is a torsion monoidal category, we will write $\mathbf{H}^0$ for the cohomological functor from $\mathbf{D}$ to its heart. We also write $L : \mathbf{D} \longrightarrow \mathbf{D}$ for the functor sending an object M in $\mathbf{D}$ to $M \otimes S$, and R for the functor sending M to $\underline{\mathrm{Hom}}(S, M)$, where S is the Tate object. By assumption, (L, R) is an adjoint pair. Let L^n and R^n denote the n-th iterations of L and R respectively. Since L is left adjoint to R, L^n is left adjoint to

R^n. Furthermore, by the Cancellation axiom (Definition 8.0.1(3)), R^nL^n is naturally isomorphic to the identity.

Since $\otimes$ is right t-exact, it induces a symmetric monoidal and a partial internal hom structure on the heart $\mathscr{C}$ of $\mathbf{D}$, which we represent by $\otimes^{\mathscr{C}}$ and $\underline{\mathrm{Hom}}_{\mathscr{C}}$. The tensor and internal hom bifunctors are given by

$$C \otimes^{\mathscr{C}} C' \stackrel{\text{def}}{=} \mathbf{H}^0(C \otimes C') \quad \text{and} \quad \underline{\mathrm{Hom}}_{\mathscr{C}}(C, C') \stackrel{\text{def}}{=} \mathbf{H}^0(\underline{\mathrm{Hom}}(C, C')).$$

Since $\underline{\mathrm{Hom}}(S, -)$ is assumed to be t-exact, Proposition 6.1.7 states that S is a semi-representable object of $\mathscr{C}$, i.e., $\underline{\mathrm{Hom}}_{\mathscr{C}}(S, -)$ is right adjoint to $- \otimes^{\mathscr{C}} S$. We let L_H and R_H denote the endofunctor on $\mathscr{C}$ given by $F \mapsto F \otimes^{\mathscr{C}} S$ and $F \mapsto \underline{\mathrm{Hom}}_{\mathscr{C}}(S, F)$ respectively. By convention, let L_H^0 and R_H^0 be the identity functor on $\mathscr{C}$, and let L_H^n and R_H^n denote the n-th iteration of L_H and R_H respectively. Since L_H is left adjoint to R_H, L_H^n is left adjoint to R_H^n for every integer $n > 0$.

Here are some additional results about the functors L_H and R_H that we will refer to throughout the remainder of this chapter. The following proposition generalizes results from Lemma 6.1.11, Proposition 6.1.14, and Proposition 6.1.15.

Proposition 8.0.2. *For all integers $n > 0$,*

1. *R_H^n is an exact functor,*
2. *there exists a natural isomorphism between R^n and R_H^n as endofunctors on* $\mathbf{HI}$*,*
3. *there exists a natural isomorphism between $\mathbf{H}^0L^n$ and L_H^n as endofunctors of* $\mathbf{HI}$*,*
4. *there exists a natural isomorphism from* id *to $R_H^nL_H^n$ as endofunctors on* $\mathbf{HI}$.

Proof. Since R is t-exact, $R_H = \mathbf{H}^0R$ is exact as an endofunctor on $\mathscr{C}$ by Proposition 6.1.6. Since composition of exact functors is exact, R_H^n is exact. This proves part (1).

To verify (2), we proceed by induction on n. The case $n = 1$ follows by definition. Now suppose $\mathbf{H}^0R^{n-1}$ is naturally isomorphic to R_H^{n-1}. Since R is t-exact, by [BBD, 1.3.17(ii)], there exists a natural isomorphism between $\mathbf{H}^0R$ and $\mathbf{H}^0R\mathbf{H}^0$. Therefore, we obtain the following chain of natural isomorphisms: $R_H^n \cong \mathbf{H}^0R\mathbf{H}^0R^{n-1} \cong \mathbf{H}^0R^n$.

Since R is t-exact, so is R^n. Furthermore, by definition of t-exactness, for all C in $\mathscr{C}$, $R^n(C)$ is an object of $\mathscr{C}$. It follows that $\mathbf{H}^0 R^n = R^n$.

For (3), since S is in $\mathscr{C}$ and $\otimes$ is right t-exact in both factors, $\mathbf{H}^0 L$ is naturally isomorphic to $\mathbf{H}^0 L \mathbf{H}^0$ as functors on $\mathscr{C}$ by [BBD, 1.3.7(ii)]. Using similar inductive arguments as in (2), we obtain a natural isomorphism between $\mathbf{H}^0 L^n$ and L^n_H.

Since the unit $\mathrm{id} \longrightarrow R^n L^n$ is a natural isomorphism in $\mathbf{D}$, $\mathbf{H}^0 \longrightarrow \mathbf{H}^0 R^n L^n$ is also a natural isomorphism. Notice that $\mathbf{H}^0$ is the identity functor on $\mathscr{C}$ and by part (2) and (3) $\mathbf{H}^0 R^n L^n$ is naturally isomorphic to $R^n_H L^n_H$. The composition

$$\mathrm{id} \longrightarrow \mathbf{H}^0 R^n L^n \longrightarrow R^n_H L^n_H$$

gives us the desired natural isomorphism, which proves part (4). □

8.1 Slice filtration of torsion monoidal categories

We begin by constructing the slice filtration on $\mathbf{D}$. This will generalize the results in Section 5.1.

Definition 8.1.1. Let $\mathbf{D}^{<n}$ be the full subcategory of $\mathbf{D}$ consisting of the objects M in $\mathbf{D}$ for which $R^n M = 0$. Let $\mathbf{D}^{\geq n}$ be the subcategory of objects M such that $M = L^n(M')$ for some M' in $\mathbf{D}$. Let $\nu^{\geq n}$ be the functor $L^n R^n$.

Notice that the arguments in the proof of [HK06, 1.1] rely only on the Cancellation axiom of $\mathbf{DM}^{\mathrm{eff},-}$, which is fulfilled by Definition 8.0.1(3) of $\mathbf{D}$. Therefore, the proof of *loc. cit.* generalizes to show that $\nu^{\geq n}$ is right adjoint to the inclusion of $\mathbf{D}^{\geq n}$ into $\mathbf{D}$.

Let M be an object of $\mathbf{D}$, and let $\eta^n : \nu^{\geq n} M \longrightarrow M$ be the counit. Complete η^n to a triangle:

$$\nu^{\geq n} M \longrightarrow M \longrightarrow M' \longrightarrow \nu^{\geq n} M'[1].$$

Copying the proof of [HK06, 1.3], we see that M' is uniquely determined up to unique isomorphism, and $M \mapsto M'$ defines a triangulated endofunctor $\nu^{<n}$ that is left adjoint to the inclusion of $\mathbf{D}^{<n}$ in $\mathbf{D}$. Finally, the discussion in the paragraphs preceding Proposition 5.1.5 can be adapted to this more general setting to show that $\nu^{\geq n}$ restricted to $\mathbf{D}^{\geq n}$ is naturally isomorphic to the identity, and $\nu^{<n}$ restricted to $\mathbf{D}^{<n}$ is also

naturally isomorphic to the identity. We have just verified the following theorem, which is a generalization of Proposition 5.1.5:

Theorem 8.1.2. *If* $\mathbf{D}$ *is a torsion monoidal category, then there exists an* $\mathbb{N}$*-indexed ascending weak filtration* $(\mathbf{D}^{<*}, \nu^{<*})$ *given by*

$$0 = \mathbf{D}^{<0} \subseteq \cdots \subseteq \mathbf{D}^{<n} \subseteq \mathbf{D}^{<n+1} \subseteq \cdots$$

and a descending weak filtration $(\mathbf{D}^{\geq *}, \nu^{\geq *})$ *given by*

$$\mathbf{D} = \mathbf{D}^{\geq 0} \supseteq \cdots \supseteq \mathbf{D}^{\geq n} \supseteq \mathbf{D}^{\geq n+1} \supseteq \cdots$$

It is possible that the weak filtrations are degenerate. However, as the following result shows, the filtration being degenerate is related to the invertibility of S. Recall that S is invertible if there exists an object T in $\mathbf{D}$ such that $T \otimes S = \mathbb{1}$.

Proposition 8.1.3. *The following are equivalent:*

1. *the filtration* $(\mathbf{D}^{<*}, \nu^{<*})$ *is trivial, i.e., each* $\mathbf{D}^{<n}$ *is zero.*

2. *the filtration* $(\mathbf{D}^{\geq *}, \nu^{\geq *})$ *is degenerate with* $\mathbf{D}^{\geq n} = \mathbf{D}$ *for all* n*.*

3. *the Tate object is invertible in* $\mathbf{D}$*.*

Proof. We first show that (1) is equivalent to (2). To see that (1) implies (2), suppose $\mathbf{D}^{<n} = 0$ for all n. We need to show that every M in $\mathbf{D}$ is isomorphic to $L^n M'$ for some M' in $\mathbf{D}$. However, for every M in $\mathbf{D}$, the following is a distinguished triangle:

$$\nu^{\geq n} M \longrightarrow M \longrightarrow \nu^{<n} M \longrightarrow \nu^{\geq n} M[1]$$

where $\nu^{<n} M$ is in $\mathbf{D}^{<n}$. But the assumption that $\mathbf{D}^{<n} = 0$ implies that $\nu^{<n} M = 0$. Therefore, $L^n R^n M \cong M$. It follows that M is in $\mathbf{D}^{\geq n}$, and $\mathbf{D}^{\geq n} = \mathbf{D}$ as desired. Conversely, if $\mathbf{D}^{\geq n} = \mathbf{D}$ then for every M in $\mathbf{D}$, $M \cong L^n M'$ for some M'. Suppose M is in $\mathbf{D}^{<n}$, then by definition $0 = R^n M = R^n L^n M' \cong M'$. Therefore, $M = L^n 0 = 0$, and $\mathbf{D}^{<n} = 0$.

Now we show that (2) is equivalent to (3). Indeed, if S is invertible with inverse T, then $M = S^{\otimes n} \otimes T^{\otimes n} \otimes M = L^n(T^n \otimes M)$ which is an object of $\mathbf{D}^{\geq n}$. Conversely, if $\mathbf{D}^{\geq n} = \mathbf{D}$, then, in particular, $\mathbf{D}^{\geq 1} = \mathbf{D}$. This implies that the unit object $\mathbb{1}$ is an object of $\mathbf{D}^{\geq 1}$. In other words, $\mathbb{1} = T \otimes S$ for some T, which shows that S is invertible. □

8.2 Torsion filtration on the heart

Let us now focus on the heart $\mathscr{C}$ of $\mathbf{D}$. In this section, we will generalize the results developed in Section 6.2 for $\mathbf{HI}$. Recall from Proposition 8.0.2 and preceding paragraphs that the endofunctors $L^n_H = \mathbf{H}^0 L^n$ and $R^n_H = \mathbf{H}^0 R^n$ are adjoint. For F in $\mathscr{C}$, let $\varphi^{<n}F$ denote the cokernel of the counit map $L^n_H R^n_H F \longrightarrow F$. Since the counit is natural in F, $F \mapsto \varphi^{<n}F$ defines an endofunctor of $\mathscr{C}$. Let $\mathscr{C}^{<n}$ be the full subcategory of all objects C in $\mathscr{C}$ with $\varphi^{<n}(C) = C$, and let $\mathscr{C}^{\geq n}$ be the full subcategory of all objects C in $\mathscr{C}$ with $\varphi^{<n}(C) = 0$. The arguments for Theorem 6.2.10 go through to give us the following result.

Theorem 8.2.1. *The functors $\varphi^{<n}, n = 1, 2 \ldots,$ define a sequence of coradicals, whose associated torsion theories $(\mathscr{T}_n, \mathscr{F}_n) = (\mathscr{C}^{\geq n}, \mathscr{C}^{<n})$ fit together to define a strong ascending cofiltration of $\mathscr{C}$:*

$$0 = \mathscr{C}^{<0} \subseteq \cdots \subseteq \mathscr{C}^{<n} \subseteq \mathscr{C}^{<n+1} \subseteq \cdots$$

and a strong descending filtration of $\mathscr{C}$:

$$\mathscr{C} = \mathscr{C}^{\geq 0} \supseteq \cdots \supseteq \mathscr{C}^{\geq n} \supseteq \mathscr{C}^{\geq n+1} \supseteq \cdots .$$

Following Definition 6.2.18, we define $\varphi^{\geq n}$ to be the kernel of the natural surjection $\mathrm{id} \longrightarrow \varphi^{<n}$. By Proposition 2.1.8 and Corollary 2.2.7, $\varphi^{\geq n}$ is an idempotent pre-radical, and is right adjoint to the inclusion of $\mathscr{C}^{\geq n}$ in $\mathscr{C}$. Furthermore, an object F is in $\mathscr{C}^{\geq n}$ if and only if $\varphi^{\geq n}F = F$.

As in the case for $\mathbf{HI}$, we can define $\mathscr{C}(n)$ to be the full subcategory of objects F such that $F \cong L^n_H F'$ for some F' in $\mathscr{C}$. As defined, $\mathscr{C}(n) \subseteq \mathscr{C}(m)$ if $n < m$. The arguments of Proposition 6.2.3 go through to give us the following proposition:

Proposition 8.2.2. *The tower of full subcategories*

$$\mathscr{C} = \mathscr{C}(0) \supseteq \cdots \supseteq \mathscr{C}(n-1) \supseteq \mathscr{C}(n) \supseteq \cdots$$

defines a weak filtration on $\mathscr{C}$.

Drawing on the analogy with **HI**, the coreflection functor from $\mathscr{C}$ to $\mathscr{C}(n)$ is given by $F \mapsto L^n_H R^n_H F$. We also have the following relationship between $\mathscr{C}(n)$ and $\mathscr{C}^{\geq n}$:

Corollary 8.2.3. *For F in $\mathscr{C}(n)$, $\mathrm{Hom}_{\mathscr{C}}(F, G) = 0$ for all G in $\mathscr{C}^{<n}$. In particular, $\mathscr{C}(n)$ is a full subcategory of $\mathscr{C}^{\geq n}$.*

Proof. If F is an object of $\mathscr{C}(n)$, then $F = L^n_H F'$ for some F' in $\mathscr{C}$. Since $R^n_H G = 0$ for all G in $\mathscr{C}^{<n}$,

$$\mathrm{Hom}_{\mathscr{C}}(F, G) = \mathrm{Hom}_{\mathscr{C}}(L^n_H F', G) = \mathrm{Hom}_{\mathscr{C}}(F', R^n_H G) = 0$$

for all G in $\mathscr{C}^{<n}$. The first statement of the corollary is now proven. The second statement follows from the definition of $\mathscr{C}^{\geq n}$ as the torsion subcategory of $\mathscr{C}^{<n}$. □

The filtrations on $\mathscr{C}$ may also be trivial. Proposition 8.2.6 and Corollary 8.2.7 below show that, as in the case for $\mathbf{D}$, the degeneracy of the filtrations are related to the the invertibility of S. Let us first consider the following lemma. Recall from Definition 8.1.1 that for a given n, $\mathbf{D}^{\geq n}$ is the full subcategory of $\mathbf{D}$ whose objects are the objects M in $\mathbf{D}$ such that $M \cong L^n M'$ for some M' in $\mathbf{D}$, and $\mathbf{D}^{<n}$ is the full subcategory of $\mathbf{D}$ whose objects are the objects M in $\mathbf{D}$ such that $R^n M = 0$.

Lemma 8.2.4. *If $\mathbf{D}^{<n} = 0$ then $\mathscr{C}^{<n} = 0$.*

Proof. Recall from Proposition 8.0.2(2) that R^n_H is naturally isomorphic to R^n on $\mathscr{C}$. Therefore, if $R^n_H C = 0$, then $R^n C = 0$. Hence, $\mathscr{C}^{<n} \subset \mathbf{D}^{\geq n}$. □

If S is invertible in $\mathbf{D}$, then by Proposition 8.1.3, $\mathbf{D}^{<n} = 0$ for all n, and by the preceding lemma, $\mathscr{C}^{<n} = 0$ for all n. As we will see in Proposition 8.2.6, $\mathscr{C}^{<n} = 0$ for all n implies $\mathscr{C}^{\geq n} = \mathscr{C}$ for all n. This shows that if S is invertible in $\mathbf{D}$, then the strong filtration and cofiltration are degenerate. However, the converse does not necessarily hold. Rather, the converse is related to a weaker condition.

Definition 8.2.5. We say that S is $\mathscr{C}$*-invertible* if there exists some T in $\mathscr{C}$ such that $T \otimes^{\mathscr{C}} S = \mathbb{1}$.

Proposition 8.2.6. *The following are equivalent:*

1. *$\mathscr{C}(*)$ is degenerate with $\mathscr{C}(n) = \mathscr{C}$ for all n,*

2. *$\mathscr{C}^{<*}$ is trivial, for all n,*

3. *$\mathscr{C}^{\geq *}$ is degenerate with $\mathscr{C}^{\geq n} = \mathscr{C}$ for all n,*

4. *S is $\mathscr{C}$-invertible.*

Proof. We first show that (1), (2), and (4) are equivalent. The proof that (1) and (4) are equivalent is the same as the proof that (2) and (3) of Proposition 8.1.3 are equivalent. To see that (2) implies (1), let F be an object in $\mathscr{C}$, and let K be the kernel of the counit $L^n_H R^n_H F \longrightarrow F$. We have the following exact sequence:

$$0 \longrightarrow K \longrightarrow L^n_H R^n_H F \longrightarrow F \longrightarrow \varphi^{<n} F \longrightarrow 0.$$

By Proposition 8.0.2(1), R^n_H is exact. Applying R^n_H, we obtain the following exact sequence:

$$0 \longrightarrow R^n_H K \longrightarrow R^n_H L^n_H R^n_H F \longrightarrow R^n_H F \longrightarrow R^n_H \varphi^{<n} F \longrightarrow 0.$$

But since $R^n_H L^n_H \cong \mathrm{id}$, $R^n_H L^n_H R^n_H F \longrightarrow R^n_H F$ is an isomorphism. Therefore, $R^n_H K = R^n_H \varphi^{<n} F = 0$. That is, K and $\varphi^{<n} F$ are in $\mathscr{C}^{<n}$. It follows that $K = \varphi^{<n} F = 0$, and therefore $F = L^n_H R^n_H F$. It follows that $\mathscr{C}(n) = \mathscr{C}$.

To show that (4) implies (2), suppose F is in $\mathscr{C}^{<n}$. Then by (4), $F \cong S^{\otimes n} \otimes^{\mathscr{C}} T^{\otimes n}$ for some T in $\mathscr{C}$. Therefore, $L^n_H R^n_H F \cong F$. However, this means that $\varphi^{<n} F = 0$. By Theorem 2.2.6, $\varphi^{<n} F = F$. Therefore, $F = 0$ and $\mathscr{C}^{<n} = 0$.

To show that (3) is equivalent to the rest, we first show that (2) implies (3). Suppose $\mathscr{C}^{<n}$ is trivial. Since for all F, $\varphi^{<n} F$ is an object of $\mathscr{C}^{<n}$, it follows that $\varphi^{<n} F = 0$. Therefore, $\varphi^{\geq n} F = F$. Thus, $\mathscr{C}^{\geq n} = \mathscr{C}$, as desired.

Finally, to show that (3) implies (2), suppose $\mathscr{C}^{\geq n} = \mathscr{C}$. By Proposition 2.2.3, $\mathscr{C}^{\geq n} \cap \mathscr{C}^{<n} = 0$. Hence, $\mathscr{C}^{<n} = 0$, as desired. $\square$

The following corollary is a direct direct consequence of Lemma 8.2.4 and Proposition 8.2.6.

Corollary 8.2.7. *If S is invertible in $\mathbf{D}$, then S is $\mathscr{C}$-invertible.*

8.3 Slice filtration on the localization of D by S

Next, for a torsion monoidal category $\mathbf{D}$, we can form the localization $\mathbf{D}[S^{-1}]$ of $\mathbf{D}$ by S (see [MVW, 8A]). The objects of $\mathbf{D}[S^{-1}]$ are pairs (M, n), where M is in $\mathbf{D}$, and n is some integer, and $(M, n+1) \cong (LM, n)$ for all M and n. Morphisms between (M, n) and (M', n') are elements of the direct limit $\varinjlim_k \mathrm{Hom}(M(n+k), M'(n'+k))$. The relationship between $\mathbf{D}$ and $\mathbf{D}[S^{-1}]$ is analogous to the relationship between $\mathbf{DM}^{\mathrm{eff},-}$ and $\mathbf{DM}$. In particular, by the Cancellation axiom (Definition 8.0.1(3)) the localization functor $\Sigma^\infty : \mathbf{D} \longrightarrow \mathbf{D}[S^{-1}]$ which sends an object M in $\mathbf{D}$ to the object $(M, 0)$ is fully faithful. Therefore, we can identify $\mathbf{D}$ as a full subcategory of $\mathbf{D}[S^{-1}]$.

There is also a tensor product on $\mathbf{D}[S^{-1}]$, given by

$$(M, n) \otimes (M', n') = (M \otimes M', n + n')$$

(see [MVW, 8A]). In the case that the cyclic permutation of $(S, 0)^{\otimes 3}$ is the identity in $\mathbf{D}[S^{-1}]$, by [MVW, 8A.10, 8A.11] the tensor product is a triangulated symmetric tensor on $\mathbf{D}[S^{-1}]$. In this case, $\mathbf{D}[S^{-1}]$ is also a torsion monoidal category. However, since S is invertible in $\mathbf{D}[S^{-1}]$, defining the weak filtrations as we have done in Section 8.1 will result in trivial weak filtrations, as we have shown in Proposition 8.1.3. Nonetheless, we can still construct weak filtrations on $\mathbf{D}[S^{-1}]$ as follows.

Definition 8.3.1. Let $\mathbf{D}[S^{-1}]^{\geq n}$ be the full subcategory of $\mathbf{D}[S^{-1}]$ with objects (M, k) such that $(M, k) \cong (M', n)$ for some M' in $\mathbf{D}$. Since $(M, n+1) \cong (LM, n)$, we have the following descending tower of full subcategories:

$$\mathbf{D}[S^{-1}] \supseteq \cdots \supseteq \mathbf{D}[S^{-1}]^{\geq 0} \supseteq \cdots \supseteq \mathbf{D}[S^{-1}]^{\geq n} \supseteq \mathbf{D}[S^{-1}]^{\geq n+1} \supseteq \cdots .$$

To show that the nested sequence of subcategories is a descending weak filtration, we need to show that for each integer n, there exists a coreflection $\nu^{\geq n} : \mathbf{D}[S^{-1}] \longrightarrow \mathbf{D}[S^{-1}]^{\geq n}$. Copying the definition of the functor $\nu^{\geq n}$ on $\mathbf{DM}$ as given in Definition 5.3.4, we define $\nu^{\geq k}$ by setting

$$\nu^{\geq k}(M, n) \stackrel{\text{def}}{=} \begin{cases} (\nu^{\geq k-n} M, n) & \text{if } k > n \\ (M, n) & \text{otherwise.} \end{cases}$$

Copying the proof of Proposition 5.3.6, we see that $\nu^{\geq k}$ is right adjoint to the inclusion of $\mathbf{D}[S^{-1}]^{\geq k}$ into $\mathbf{D}[S^{-1}]$. Furthermore, the restriction of $\nu^{\geq k}$ to $\mathbf{D}[S^{-1}]^{\geq k}$ is the identity. This shows that $(\mathbf{D}[S^{-1}]^{\geq *}, \nu^{\geq *})$ is a descending weak filtration of $\mathbf{D}[S^{-1}]$.

Remark 8.3.2. Since $(M, 0) \cong (M', n)$ if and only if $M \cong L^n M'$, the image of $\mathbf{D}^{\geq n}$ under Σ^∞ is precisely $\mathbf{D}[S^{-1}]^{\geq n}$. In particular, we can identify $\mathbf{D}$ with the full subcategory $\mathbf{D}[S^{-1}]^{\geq 0}$, and the preceding discussion shows that $\nu^{\geq 0}$ is a right adjoint to Σ^∞.

Next, let $\mathbf{D}[S^{-1}]^{<n}$ be the full subcategory of objects (M, k) where $\nu^{\geq n}(M, k) = 0$. Since $\nu^{\geq k}(M, n) = 0$ implies that $\nu^{\geq k+1}(M, n) = 0$, we obtain the following ascending tower of full subcategories of $\mathbf{D}[S^{-1}]$:

$$\cdots \subseteq \mathbf{D}[S^{-1}]^{<0} \subseteq \cdots \subseteq \mathbf{D}[S^{-1}]^{<n} \subseteq \mathbf{D}[S^{-1}]^{<n+1} \subseteq \cdots \mathbf{D}[S^{-1}].$$

We want to show that this tower of full subcategories defines a weak filtration of $\mathbf{D}[S^{-1}]$ by showing that, for each n, there exists a reflection $\nu^{<n} : \mathbf{D}[S^{-1}] \longrightarrow \mathbf{D}[S^{-1}]^{<n}$. Copying the definition of the functor $\nu^{<k}$ on $\mathbf{DM}$ as given in Definition 5.3.11, we define $\nu^{<k}$ by setting:

$$\nu^{<k}(M, n) = \begin{cases} (\nu^{<n-k} M, n) & \text{if } n > k \\ 0 & \text{otherwise.} \end{cases}$$

The arguments for [HK06, 1.3(i)] go through in this general setting to show that for each k, $\nu^{<k}$ is a triangulated functor that is right adjoint to the inclusion of $\mathbf{D}[S^{-1}]^{<k}$ into $\mathbf{D}[S^{-1}]$. Moreover, the restriction of $\nu^{<k}$ to $\mathbf{D}[S^{-1}]^{<k}$ is naturally isomorphic to the identity (*cf.* Proposition 5.3.12). We have just proved the following theorem, which generalizes the results in Section 5.3.

Theorem 8.3.3. *The category of* $\mathbf{D}[S^{-1}]$ *is equipped with a descending weak filtration given by* $(\mathbf{D}[S^{-1}]^{\geq *}, \varphi^{\geq *})$ *and an ascending weak filtration given by* $(\mathbf{D}[S^{-1}]^{<*}, \varphi^{<*})$.

The following proposition, which is a consequence of Proposition 8.1.3, relate the degeneracy of the weak filtrations that we defined above with the invertibility of the Tate object S.

Proposition 8.3.4. *The following are equivalent:*

1. *the categories* $\mathbf{D}[S^{-1}]^{<n}$ *are trivial,*

2. *the categories* $\mathbf{D}[S^{-1}]^{\geq n}$ *is degenerate with* $\mathbf{D}[S^{-1}]^{\geq n} = \mathbf{D}[S^{-1}]$ *for all* n*,*

3. S *is invertible in* $\mathbf{D}$*.*

Proof. If S is invertible, then $\mathbf{D}[S^{-1}]$ is equivalent to $\mathbf{D}$. The fact (3) implies (1) and (2) follows directly from Proposition 8.1.3.

To show that (1) implies (2), suppose $\mathbf{D}[S^{-1}]^{<n} = 0$ for all n. For any object (M, k) of $\mathbf{D}[S^{-1}]$ and any integer n, we have the following distinguished triangle

$$\nu^{\geq n}(M, k) \longrightarrow (M, k) \longrightarrow \nu^{<n}(M, k) \longrightarrow \nu^{\geq n}(M, k)[1].$$

Notice that $\nu^{<n}(M, k)$ is an object of $\mathbf{D}[S^{-1}]^{<n}$ and $\nu^{\geq n}(M, k)$ is an object of $\mathbf{D}[S^{-1}]^{\geq n}$. By the assumption that $\mathbf{D}[S^{-1}]^{<n} = 0$, we see that $(M, k) \cong \nu^{\geq n}(M, k)$, and therefore (M, k) is in $\mathbf{D}[S^{-1}]^{\geq n}$. Therefore, $\mathbf{D}[S^{-1}]^{\geq n} = \mathbf{D}[S^{-1}]$ for all n.

To show that (2) implies (3), suppose $\mathbf{D}[S^{-1}]^{\geq n} = \mathbf{D}[S^{-1}]$ for all n. In particular, $\mathbf{D}[S^{-1}]^{\geq 1} = \mathbf{D}[S^{-1}]$. This implies that $(\mathbb{1}, 0)$ is an object of $\mathbf{D}[S^{-1}]^{\geq 1}$. By definition of $\mathbf{D}[S^{-1}]^{\geq 1}$, $(\mathbb{1}, 0) \cong (T, 1)$ for some T. Therefore, $\mathbb{1} \cong LT = S \otimes T$, and S is invertible in $\mathbf{D}$. □

8.4 Torsion Filtration on the Stable Localization of $\mathscr{C}$ by S

In this section, we generalize the results of Section 7.1. Copying the construction of $\mathbf{HI}_*$, we define the stable localization of $\mathscr{C}$ as follows:

Definition 8.4.1. Let $\mathscr{C}_S$ denote the category whose objects are the $\mathbb{Z}$-graded objects C_* in $\mathscr{C}$ together with a map $s_n : L_H C_n \longrightarrow C_{n+1}$ for each integer n such that the the corresponding adjunction map $w_n : C_{n+1} \longrightarrow R_H C_n$ is an isomorphism. Following Section 7.1, we call s_n the *n-th suspension map*, and w_n the *n-th delooping map*. We will represent an object of $\mathscr{C}_S$ by (C_*, w_*) or simply C_* if the collection of delooping maps are clear.

By the Cancellation axiom, $C \mapsto L_H C$ is fully faithful, and therefore, the arguments in [Dég10, 1.8] go through for $\mathscr{C}_S$ to show that there is a fully faithful functor σ^∞ from $\mathscr{C}$ to $\mathscr{C}_S$ given by $C \mapsto (C_*, w_*)$, where the n-th graded component of C_* is given by

$$C_n \stackrel{\text{def}}{=} \begin{cases} L_H^n C & \text{if } n > 0 \\ C & \text{if } n = 0 \\ R_H^{|n|} C & \text{otherwise,} \end{cases}$$

and the n-th delooping $w_n : C_n \longrightarrow R_H(C_{n+1})$ is the unit map for $n \geq 0$ and the identity for $n < 0$. As in the case for $\mathbf{HI}_*$, σ^∞ has a right adjoint $\omega^\infty : \mathscr{C} \longrightarrow \mathscr{C}_S$, given by $(C_*, w_*) \mapsto C_0$. Thus, we can view $\mathscr{C}$ as a full coreflective subcategory of $\mathscr{C}_S$ whose objects are the (C_*, w_*) such that $C_n = L_H^n C_0$ for all $n > 0$.

Definition 8.4.2. Copying the definition of $\varphi^{<n}$ in Definition 7.1.6, for an object (C_*, w_*) of $\mathscr{C}_S$, we define $\varphi^{<n}(C_*)$ to be the object in $\mathscr{C}_S$ where

$$(\varphi^{<n}(C_*))_k \stackrel{\text{def}}{=} \begin{cases} \varphi^{<n+k}(C_k) & \text{if } n + k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For ease of notation, we will write $\varphi_k^{<n}(C_*)$ for the k-th graded component of $\varphi^{<n}(C_*)$.

As the arguments of Theorem 7.1.9 are entirely formal, replacing $\mathbf{HI}_*$ by $\mathscr{C}_S$ and $\varphi_*^{<k}$ by $\varphi^{<k}$, we obtain the following proposition which is needed in the construction of the strong filtration and cofiltration on $\mathscr{C}_S$.

Proposition 8.4.3. *For each integer n, $\varphi^{<n}$ is a coradical of the category $\mathscr{C}_S$.*

We can now define the full subcategories in the strong filtration and cofiltration of $\mathscr{C}_S$. Recall from Theorem 2.2.6 that if φ is a coradical, then the torsion subcategory of φ is the full subcategory $\mathscr{T}$ consisting of the objects T such that $\varphi(T) = 0$, and the torsionfree subcategory of φ is the full subcategory $\mathscr{F}$ whose objects are the objects F such that the natural map $F \longrightarrow \varphi(F)$ is an isomorphism.

Definition 8.4.4. Let $\mathscr{C}_S^{<n}$ be the torsionfree subcategory of $\varphi^{<n}$, i.e., C_* is an object of $\mathscr{C}_S^{<n}$ if and only if $\varphi^{<n}(C_*) = C_*$. Let $\mathscr{C}_S^{\geq n}$ be the torsion subcategory of $\varphi^{<n}$. The objects of $\mathscr{C}_S^{\geq n}$ are the C_* in $\mathscr{C}_S$ such that $\varphi^{<n}(C_*) = 0$.

Copying the proof for Corollary 7.1.12 we obtain the following theorem.

Theorem 8.4.5. *The sequence of functors $\varphi^{<n}, n = \ldots, -1, 0, 1, \ldots$ on $\mathscr{C}_S$ is a $\mathbb{Z}$-indexed sequence of coradicals whose associated torsion theories*

$$(\mathscr{T}_n, \mathscr{F}_n) = (\mathscr{C}_S^{\geq n}, \mathscr{C}_S^{<n})$$

define an ascending strong cofiltration

$$\cdots \subseteq \mathscr{C}_S^{<0} \subseteq \cdots \subseteq \mathscr{C}_S^{<n} \subseteq \mathscr{C}_S^{<n+1} \subseteq \cdots$$

and a strong descending filtration

$$\mathscr{C}_S \supseteq \cdots \supseteq \mathscr{C}_S^{\geq 0} \supseteq \cdots \supseteq \mathscr{C}_S^{\geq n} \supseteq \mathscr{C}_S^{\geq n+1} \supseteq \cdots .$$

on $\mathscr{C}_S$.

The following proposition shows that S is $\mathscr{C}$-invertible if and only if each of the weak filtrations above are degenerate:

Proposition 8.4.6. *The following are equivalent:*

1. *$\mathscr{C}_S^{<*}$ is trivial, for all n,*
2. *$\mathscr{C}_S^{\geq *}$ is degenerate with $\mathscr{C}_S^{\geq n} = \mathscr{C}$ for all n,*
3. *S is $\mathscr{C}$-invertible.*

Proof. To see that (1) implies (2), suppose $\mathscr{C}_S^{<n} = 0$ for all integers n. Then $\varphi^{<n}(C_*) = 0$ for all C_* in $\mathscr{C}_S$, and therefore $\varphi^{<n} = 0$ for all n. Since C_* is in $\mathscr{C}_S^{\geq n}$ if and only if $\varphi^{<n}(C_*) = 0$ (see Definition 8.4.4), it follows that $\mathscr{C}_S^{\geq n} = \mathscr{C}_S$ for all n.

Now, assume $\mathscr{C}_S^{\geq n} = \mathscr{C}_S$ for all n. By Proposition 2.2.3, $\mathscr{C}_S^{<n} \cap \mathscr{C}_S^{\geq n} = 0$. Hence, $\mathscr{C}_S^{<n} = 0$ for all n. This shows that (2) implies (1).

To show that (1) implies (3), suppose $\mathscr{C}_S^{<n} = 0$ for all n. As we have shown in the proof of (1) implies (2), $\varphi^{<n} = 0$ as an endofunctor on $\mathscr{C}_S$ for all n, which further implies that $\varphi^{<n} = 0$ as an endofunctor on $\mathscr{C}$ for all $n > 0$. This implies that $\mathscr{C}^{<n} = 0$ for all n, and by Proposition 8.2.3, S is invertible in $\mathscr{C}$.

To see that (3) implies (1), suppose S is invertible in $\mathscr{C}$. Then by Proposition 8.2.3, $\mathscr{C}^{<n} = 0$ for all n. Therefore, $\varphi^{<n}(C) = 0$ for all C in $\mathscr{C}$ and $n > 0$. Hence, $\varphi^{<n}(C_*) = 0$ for all C_* in $\mathscr{C}_S$ and integer n. It follows that the torsionfree categories are trivial, i.e., $\mathscr{C}_S^{<n} = 0$ for all integer n. □

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